3D Rotations

CS 6384 Computer Vision
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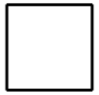
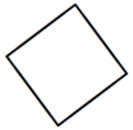
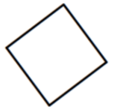
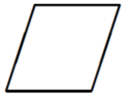
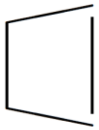
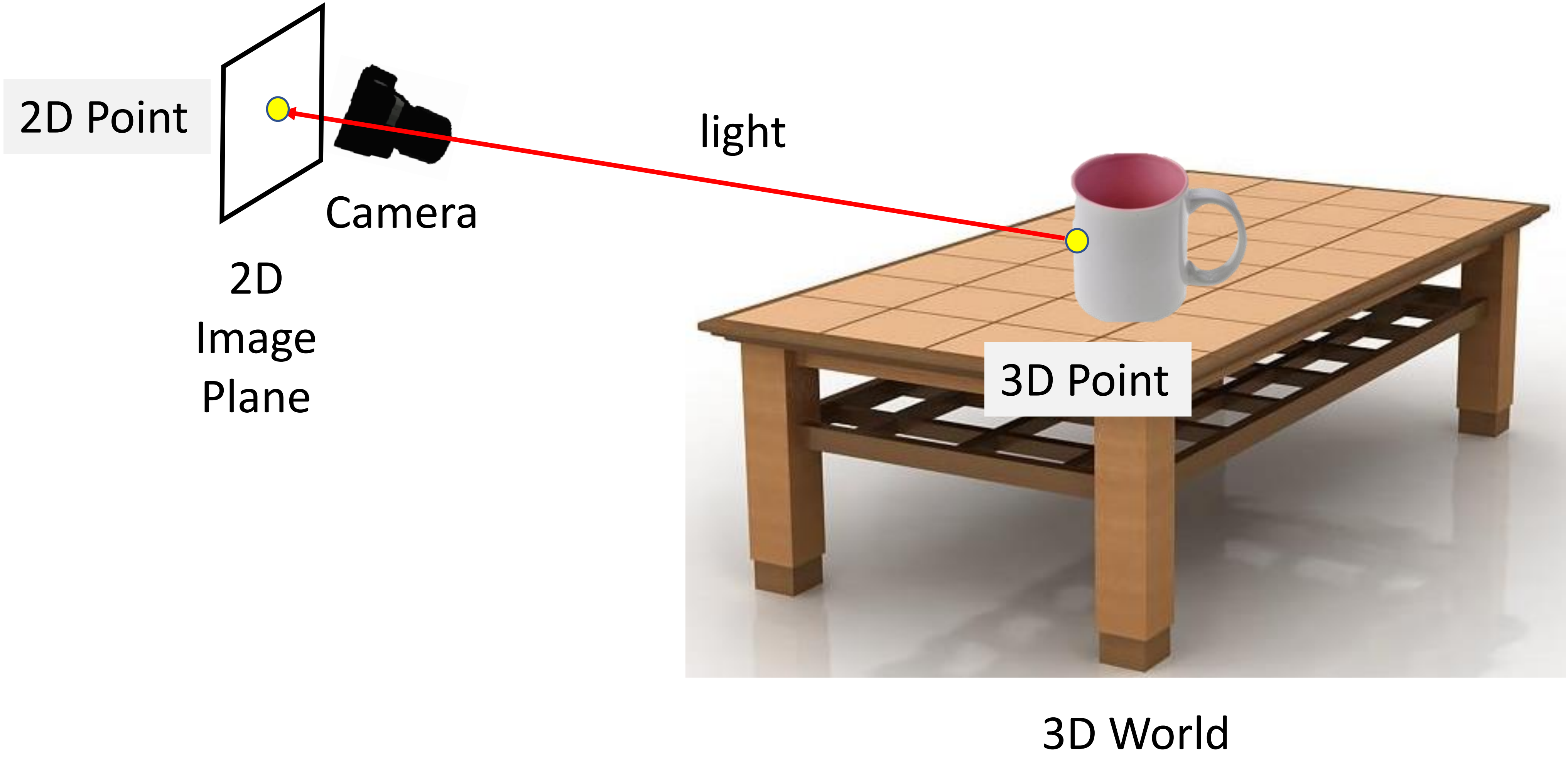
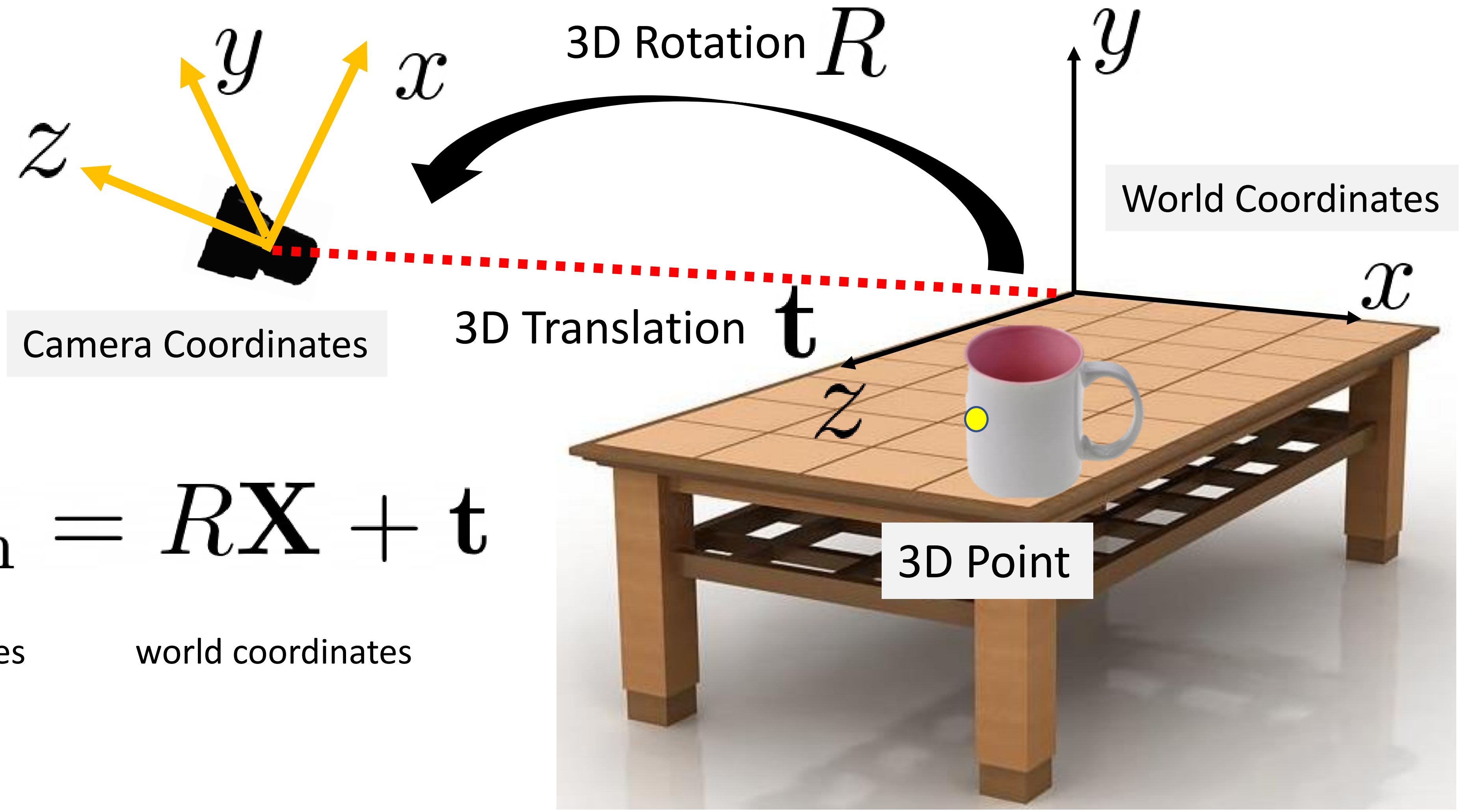
Recall 3D Transformations

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{3\times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$	7	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{3\times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{4\times 4}$	15	straight lines	

Recall Geometry in Image Generation



Camera Rotation and Translation



$$\mathbf{X}_{\mathrm{cam}} = R\mathbf{X} + \mathbf{t}$$

camera coordinates　　world coordinates

An Example for Camera Pose Tracking

ORB-SLAM

https://webdiis.unizar.es/~raulmur/orbslam/

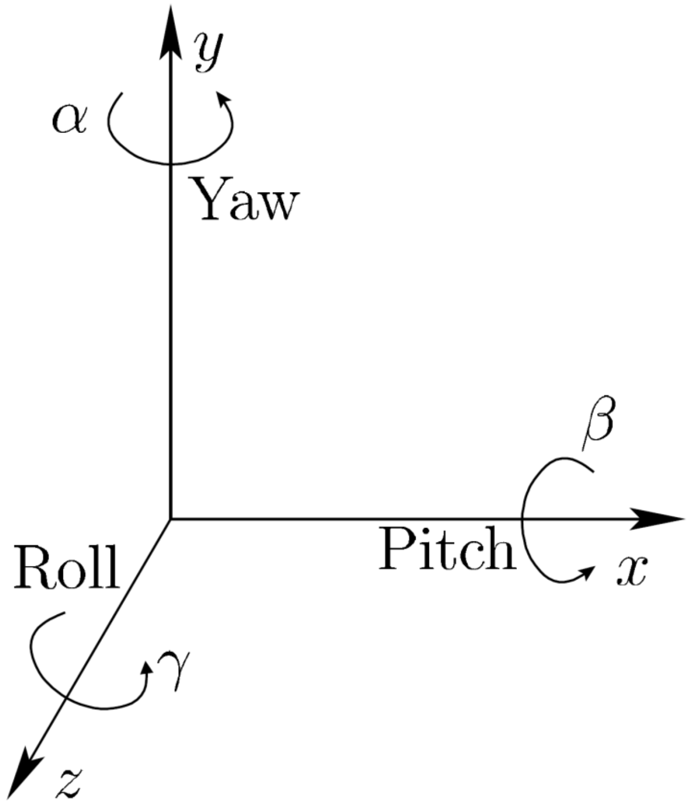
3D Rotations

- Unit-length columns
- Perpendicular columns
- $\det M = 1$
- 3 DOFs

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Euler Angles: Yaw, Pitch, Roll

- Counterclockwise rotation



Roll $$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pitch $$R_x(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

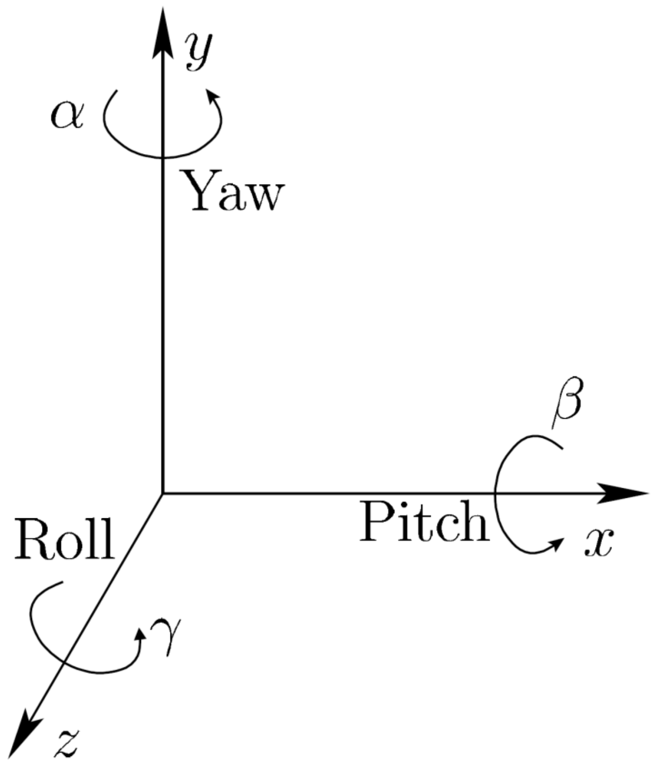
Yaw $$R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Combining Rotations

- Matrix multiplications are "backwards"

$$R(\alpha, \beta, \gamma) = R_y(\alpha) R_x(\beta) R_z(\gamma)$$

$$\alpha, \gamma \in [0, 2\pi] \qquad \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

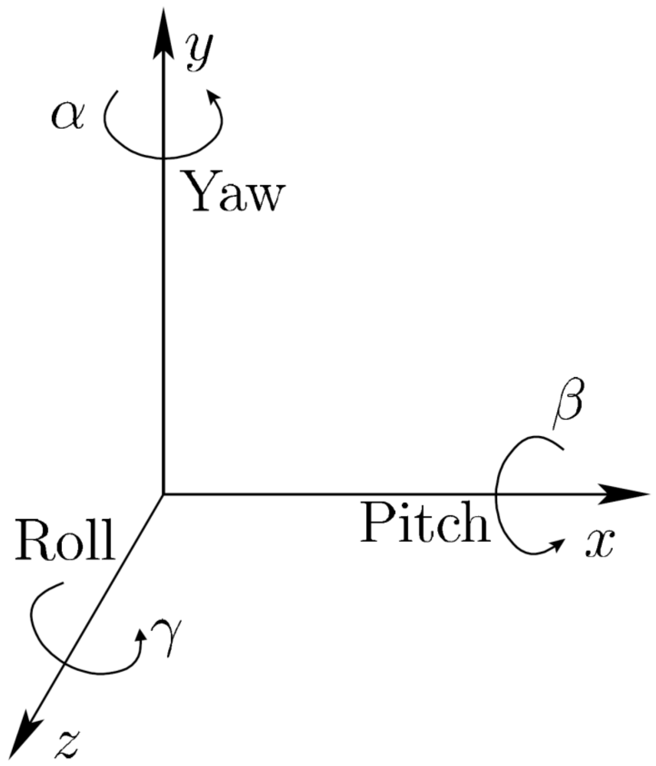


The Order Matters

- 12 possible sequences of rotation axes

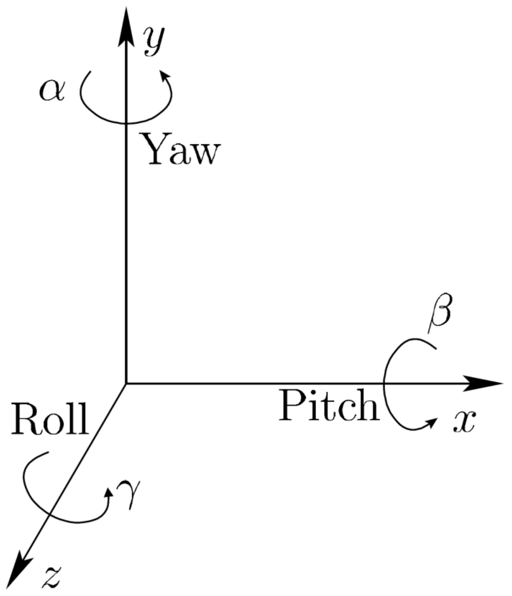
Proper Euler angles (z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y)

Tait–Bryan angles (x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z)



Kinematic Singularities

- When pitch $\beta = \frac{\pi}{2}$



$$R(\alpha, \beta, \gamma) = R_y(\alpha) R_x(\beta) R_z(\gamma)$$

$$\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha-\gamma) & \sin(\alpha-\gamma) & 0 \\ 0 & 0 & -1 \\ -\sin(\alpha-\gamma) & \cos(\alpha-\gamma) & 0 \end{bmatrix}$$

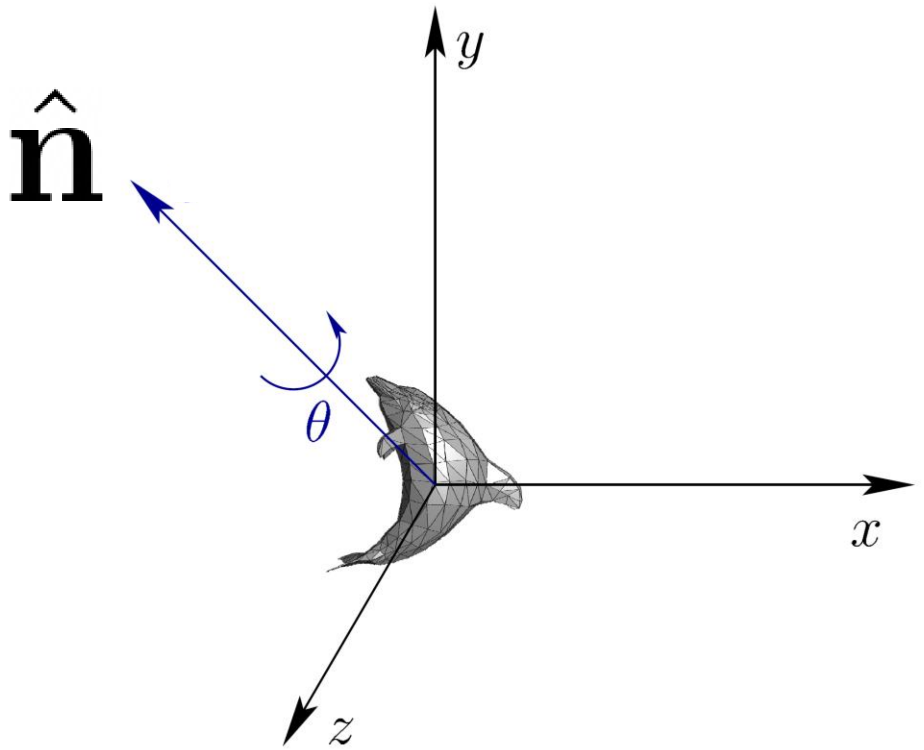
Longitude
Latitude

Only one DOF

The kinematic singularity often causes the viewpoint to spin uncontrollably in VR.

Axis-Angle Representations of Rotation

- Euler's rotation theorem: every 3D rotation can be considered as a rotation by an angle about an axis through the origin



Unit vector

$$\hat{\mathbf{n}} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)$$

$$\boldsymbol{\omega} = \theta \hat{\mathbf{n}}$$

DOFs?

Rodrigues' Rotation Formula

Axis-angle to rotation matrix $$\boldsymbol{\omega} = \theta \hat{\mathbf{n}}$$

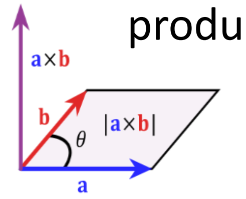
$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin\theta [\hat{\mathbf{n}}]_\times + (1 - \cos\theta)[\hat{\mathbf{n}}]_\times^2$$

Cross product matrix

Skew-symmetric Matrix

$$[\hat{\mathbf{n}}]_\times = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix}$$

Rodrigues' Rotation Formula

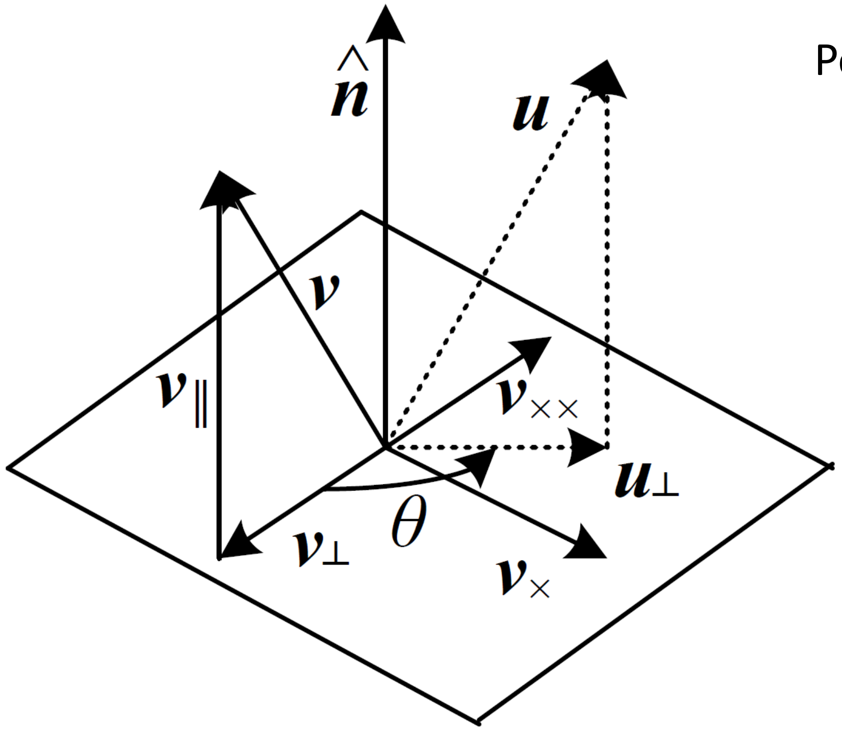


Vector cross product

Vector dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

Rotate **v** around $\hat{\mathbf{n}}$ to get **u**



$$\omega = \theta \hat{\mathbf{n}}$$

Projection $\mathbf{v}_{\parallel} = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{v}) = (\hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{v}$

Perpendicular residual $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{v}$

Cross product (rotation 90°) $\mathbf{v}_{\times} = \hat{\mathbf{n}} \times \mathbf{v}_{\perp} = \hat{\mathbf{n}} \times \mathbf{v} = [\hat{\mathbf{n}}]_{\times}\mathbf{v}$

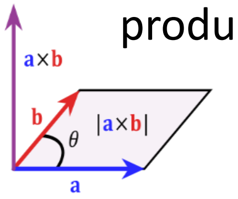
Cross product matrix
$$[\hat{\mathbf{n}}]_{\times} = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix}$$

Rotation 90° again $\mathbf{v}_{\times\times} = \hat{\mathbf{n}} \times \mathbf{v}_{\times} = [\hat{\mathbf{n}}]^2_{\times}\mathbf{v} = -\mathbf{v}_{\perp}$

$$\mathbf{v}_{\parallel} = \mathbf{v} - \mathbf{v}_{\perp} = \mathbf{v} + \mathbf{v}_{\times\times} = (\mathbf{I} + [\hat{\mathbf{n}}]^2_{\times})\mathbf{v}$$

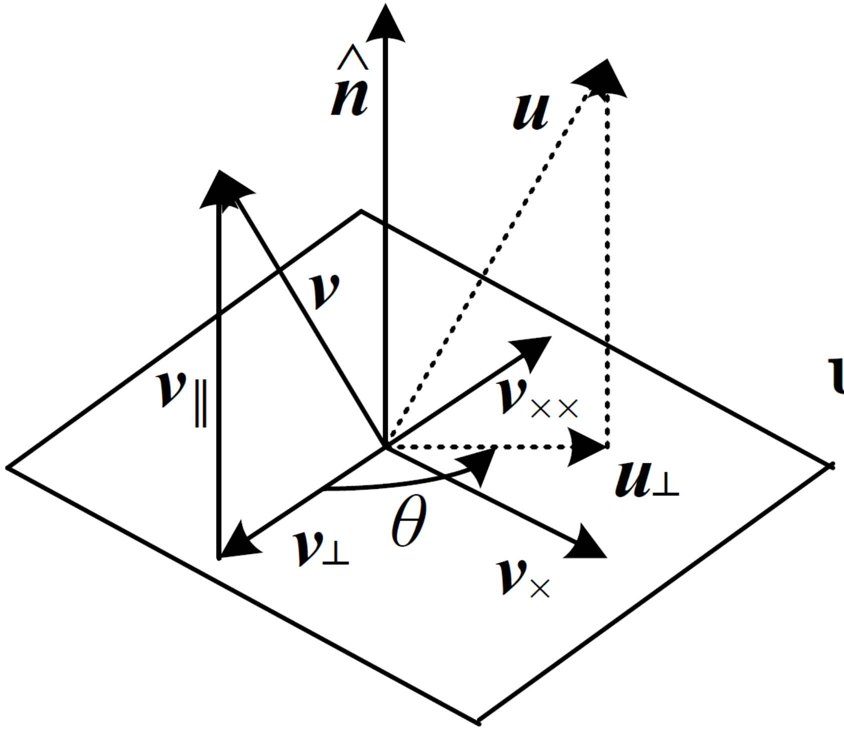
Rodrigues' Rotation Formula

Vector cross product



Vector dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$



In-plane component

$$\mathbf{u}_{\perp} = \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{v}_{\times} = (\sin\theta[\hat{\mathbf{n}}]_{\times} - \cos\theta[\hat{\mathbf{n}}]_{\times}^{2})\mathbf{v}$$

$$\mathbf{v}_{\|} = \mathbf{v} - \mathbf{v}_{\perp} = \mathbf{v} + \mathbf{v}_{\times\times} = (\mathbf{I} + [\hat{\mathbf{n}}]_{\times}^{2})\mathbf{v}$$

$$\mathbf{u} = \mathbf{u}_{\perp} + \mathbf{v}_{\|} = (\mathbf{I} + \sin\theta[\hat{\mathbf{n}}]_{\times} + (1 - \cos\theta)[\hat{\mathbf{n}}]_{\times}^{2})\mathbf{v}$$

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin\theta[\hat{\mathbf{n}}]_{\times} + (1 - \cos\theta)[\hat{\mathbf{n}}]_{\times}^{2}$$

Rodrigues' formula

$$\boldsymbol{\omega} = \theta\hat{\mathbf{n}}$$

$$[\hat{\mathbf{n}}]_{\times} = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix}$$

Rodrigues' Rotation Formula

$$\boldsymbol{\omega} = \theta\hat{\mathbf{n}} = (\omega_x, \omega_y, \omega_z)$$ Axis-angle

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin\theta[\hat{\mathbf{n}}]_\times + (1 - \cos\theta)[\hat{\mathbf{n}}]^2_\times$$ Rotation matrix

For small rotation angles $\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \frac{\theta^2}{2} \approx 1$

$$\mathbf{R}(\boldsymbol{\omega}) \approx \mathbf{I} + \sin\theta[\hat{\mathbf{n}}]_\times \approx \mathbf{I} + [\theta\hat{\mathbf{n}}]_\times = \begin{bmatrix} 1 & -\omega_z & \omega_y \\ \omega_z & 1 & -\omega_x \\ -\omega_y & \omega_x & 1 \end{bmatrix}$$ linearized relationship

$$\mathbf{R}(\boldsymbol{\omega})\mathbf{v} \approx \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v}$$

Derivative $\frac{\partial \mathbf{R}\mathbf{v}}{\partial \boldsymbol{\omega}^T} = -[\mathbf{v}]_\times = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}$

Cross prodcut $w \times v = -v \times w$

SO(n): Special Orthogonal Group

- SO(n): Space of rotation matrices in $\mathbb{R}^n$

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : RR^T = I, \det(R) = 1\}$$

- SO(3): space of 3D rotation matrices
- Group is a set G, with an operation●, satisfying the following axioms:
  - Closure: $a \in G, b \in G \Rightarrow a \cdot b \in G$
  - Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c), \forall a, b, c \in G$
  - Identity element: $\exists e \in G, e \cdot a = a, \forall a \in G$
  - Inverse element: $\forall a \in G, \exists b \in G, a \cdot b = b \cdot a = e$

Exponential Map for SO(3)

- Matrix exponential $\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$ factorial $X : n \times n$

Rodrigues' formula

$$\boldsymbol{\omega} = \theta \hat{\mathbf{n}} \qquad \mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin\theta [\hat{\mathbf{n}}]_\times + (1 - \cos\theta)[\hat{\mathbf{n}}]_\times^2$$

$$\begin{aligned}
\exp[\boldsymbol{\omega}]_\times &= \mathbf{I} + \theta[\hat{\mathbf{n}}]_\times + \frac{\theta^2}{2}[\hat{\mathbf{n}}]_\times^2 + \frac{\theta^3}{3!}[\hat{\mathbf{n}}]_\times^3 + \cdots \\
&= \mathbf{I} + (\theta - \frac{\theta^3}{3!} + \cdots)[\hat{\mathbf{n}}]_\times + (\frac{\theta^2}{2} - \frac{\theta^4}{4!} + \cdots)[\hat{\mathbf{n}}]_\times^2 \\
&= \mathbf{I} + \sin\theta[\hat{\mathbf{n}}]_\times + (1 - \cos\theta)[\hat{\mathbf{n}}]_\times^2
\end{aligned}$$

$$[\hat{\mathbf{n}}]_\times^{k+2} = -[\hat{\mathbf{n}}]_\times^k$$

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \exp[\omega]_\times$$

Matrix Logarithm of Rotations

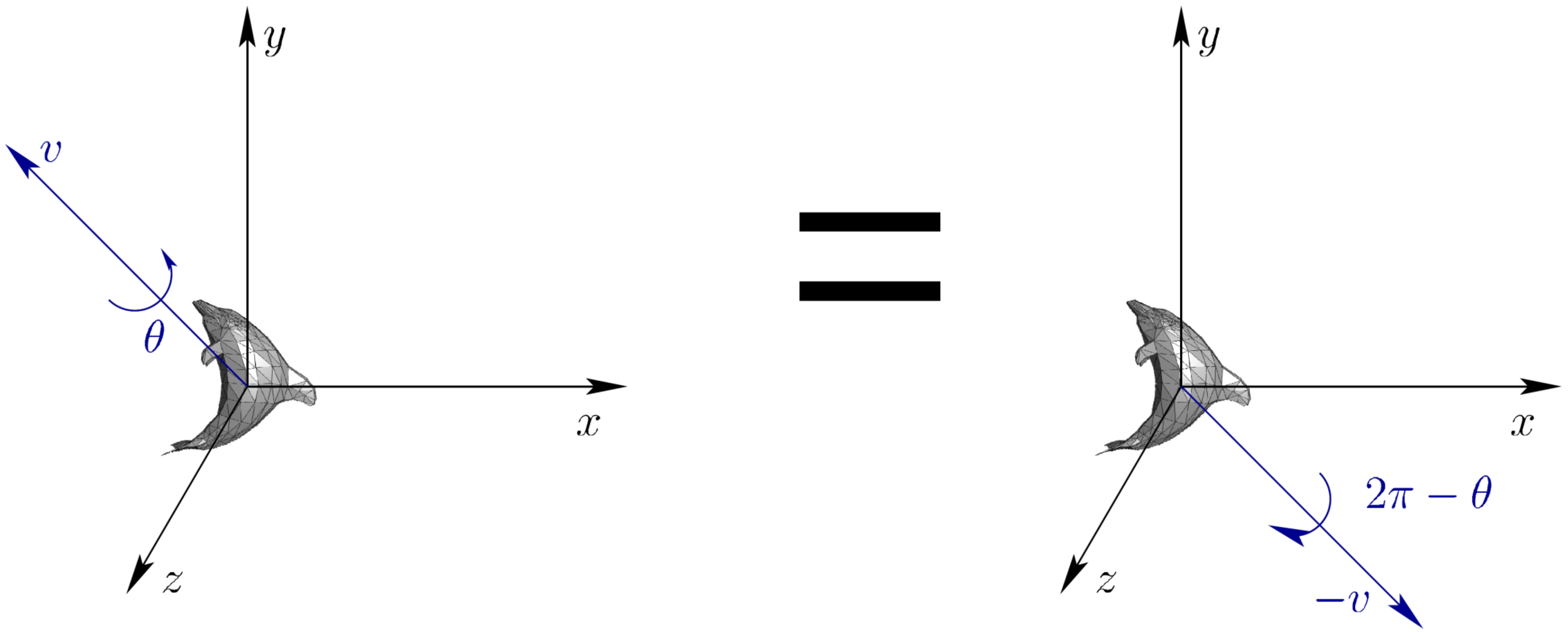
- If $\hat{\omega}\theta \in \mathbb{R}^3$ represent the exponential coordinates of rotation R, then the matrix logarithm of the rotation R is

$$[\hat{\omega}\theta] = [\hat{\omega}]\theta$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

$$\begin{aligned} \exp: \quad & [\hat{\omega}]\theta \in so(3) & \rightarrow \quad & R \in SO(3), \\ \log: \quad & R \in SO(3) & \rightarrow \quad & [\hat{\omega}]\theta \in so(3). \end{aligned}$$

Two-to-one Problem of Axis-Angle Representations



Quaternions

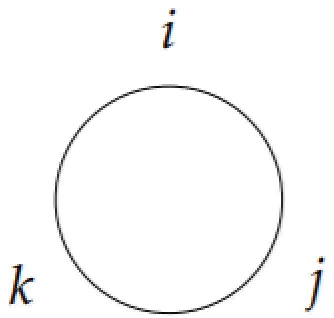
- Quaternions generalize complex numbers and can be used to represents 3D rotations

$$q = w + \underbrace{xi + yj + zk}$$

Scale (real part) Vector (imaginary part)

- Properties

$$i^2 = j^2 = k^2 = -1$$
$$ij = k, ji = -k$$
$$jk = i, kj = -i$$
$$ki = j, ik = -j$$



Quaternion Addition and Multiplication

- Addition

$$p + q = (p_0 + q_0) + (p_1 + q_1)\boldsymbol{i} + (p_2 + q_2)\boldsymbol{j} + (p_3 + q_3)\boldsymbol{k}$$

- Multiplication

$$\begin{aligned} pq &= (p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k})(q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \\ &= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + p_0(q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) + q_0(p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}) \\ &\qquad + (p_2q_3 - p_3q_2)\boldsymbol{i} + (p_3q_1 - p_1q_3)\boldsymbol{j} + (p_1q_2 - p_2q_1)\boldsymbol{k}. \end{aligned}$$

$$pq = p_0q_0 - \boldsymbol{p} \cdot \boldsymbol{q} + p_0\boldsymbol{q} + q_0\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{q}$$

$$\boldsymbol{p} = (p_1, p_2, p_3) \quad \boldsymbol{q} = (q_1, q_2, q_3)$$

Complex Conjugate, Norm and Inverse

- Conjugate $$q = q_0 + \boldsymbol{q} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$$
$$q^* = q_0 - \boldsymbol{q} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}$$

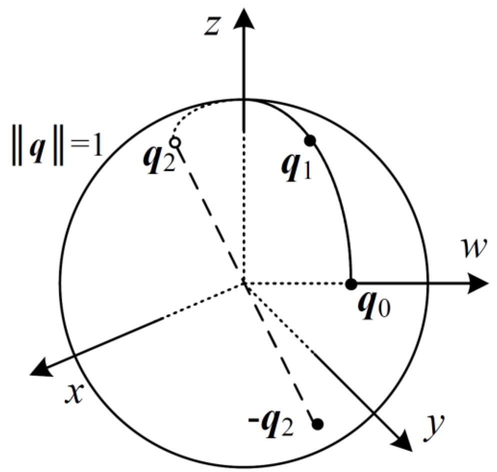
- Norm $$|q| = \sqrt{q^*q}$$
$$\begin{aligned} q^*q &= (q_0 - \boldsymbol{q})(q_0 + \boldsymbol{q}) \\ &= q_0q_0 - (-\boldsymbol{q}) \cdot \boldsymbol{q} + q_0\boldsymbol{q} + (-\boldsymbol{q})q_0 + (-\boldsymbol{q}) \times \boldsymbol{q} \\ &= q_0^2 + \boldsymbol{q} \cdot \boldsymbol{q} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \\ &= qq^*. \end{aligned}$$

- Inverse $$q^{-1} = \frac{q^*}{|q|^2} \qquad q^{-1}q = qq^{-1} = 1$$

Unit Quaternions as 3D Rotations

- For unit quaternions, axis-angle

$$q = (w, \mathbf{v}) = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\hat{\mathbf{n}})$$



- For $\boldsymbol{v} \in \mathbb{R}^3$, rotation according to a unit quaternion $q = q_0 + \boldsymbol{q}$

$$\begin{aligned} L_q(\boldsymbol{v}) &= q\boldsymbol{v}q^* \\ &= (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}\cdot\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q}\times\boldsymbol{v}) \end{aligned}$$

The real part of v is 0

Unit Quaternions as 3D Rotations

$$q = (w, \mathbf{v}) = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\hat{\mathbf{n}})$$

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin\theta[\hat{\mathbf{n}}]_\times + (1 - \cos\theta)[\hat{\mathbf{n}}]_\times^2$$
$$= \mathbf{I} + 2w[\mathbf{v}]_\times + 2[\mathbf{v}]_\times^2.$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$(1 - \cos\theta) = 2\sin^2\frac{\theta}{2}$$

$$[\mathbf{v}]_\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

$$[\mathbf{v}]_\times^2 = \begin{bmatrix} -y^2 - z^2 & xy & xz \\ xy & -x^2 - z^2 & yz \\ xz & yz & -x^2 - y^2 \end{bmatrix}$$

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - zw) & 2(xz + yw) \\ 2(xy + zw) & 1 - 2(x^2 + z^2) & 2(yz - xw) \\ 2(xz - yw) & 2(yz + xw) & 1 - 2(x^2 + y^2) \end{bmatrix}$$

Unit quaternion to rotation matrix

Unit Quaternions as 3D Rotations

- Composing rotations using unit quaternions

$$\mathbf{q}_2 = \mathbf{q}_0 \mathbf{q}_1$$

$$\mathbf{R}(\mathbf{q}_2) = \mathbf{R}(\mathbf{q}_0)\mathbf{R}(\mathbf{q}_1)$$

Two Equivalent Quaternions for 3D Rotation

- Multiply -1 to a quaternion

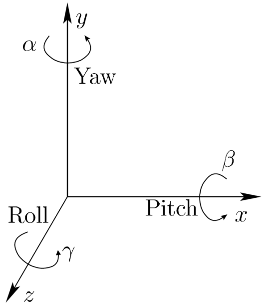
$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\frac{\vec{u}}{||\vec{u}||}$$

$$-q = \cos\left(\frac{\theta}{2} + \pi\right) + \sin\left(\frac{\theta}{2} + \pi\right)\frac{\vec{u}}{||\vec{u}||}$$

- q rotates θ , $-q$ rotates $\theta + 2\pi$

3D Rotation Representations

- Rotation matrix R
- Euler angles



- Axis-angle
  - Minimal representation $\boldsymbol{\omega} = \theta\hat{\mathbf{n}}$
- Unit quaternion
  - Continuous rotations

$$q = w + xi + yj + zk$$

Further Reading

- Section 2.1, Computer Vision, Richard Szeliski
- Quaternion and Rotations, Yan-Bin Jia, https://graphics.stanford.edu/courses/cs348a-17-winter/Papers/quaternion.pdf
- Introduction to Robotics, Prof. Wei Zhang, OSU, Lecture 3, Rotational Motion, http://www2.ece.ohio-state.edu/~zhang/RoboticsClass/index.html
- On the Continuity of Rotation Representations in Neural Networks. Zhou et al., CVPR, 2019.