# Geometric Primitives and Transformations

CS 6384 Computer Vision

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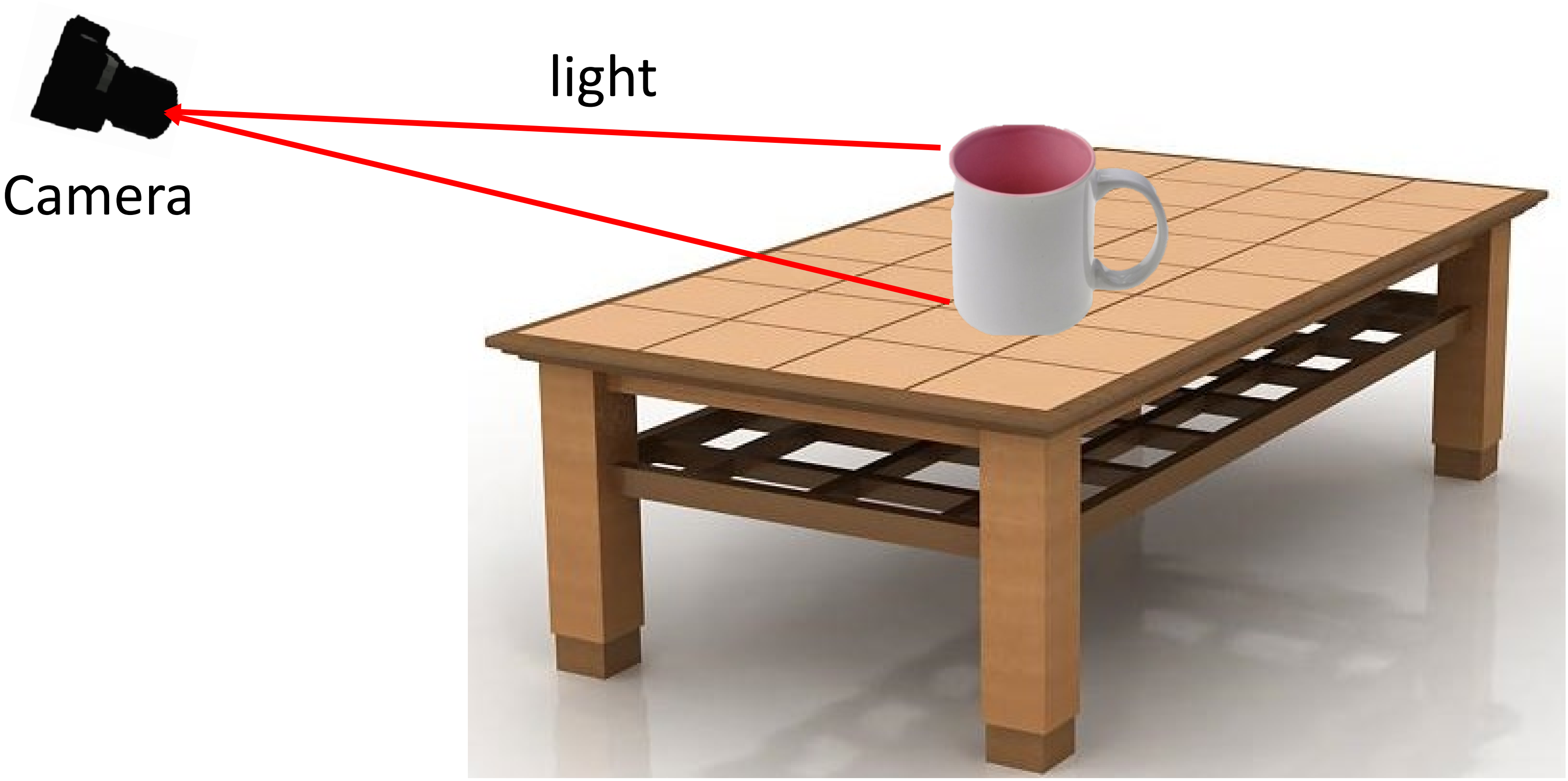
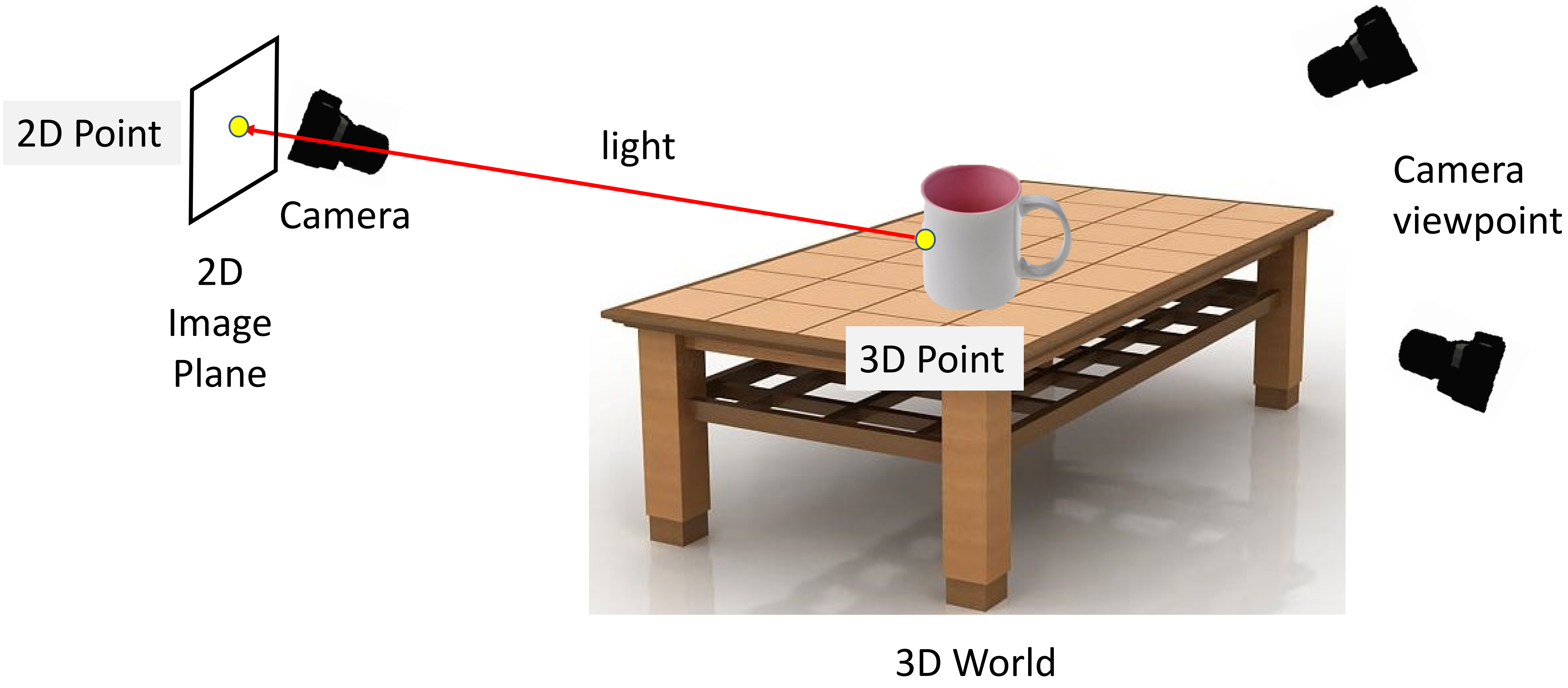
The University of Texas at Dallas

# How are Images Generated?

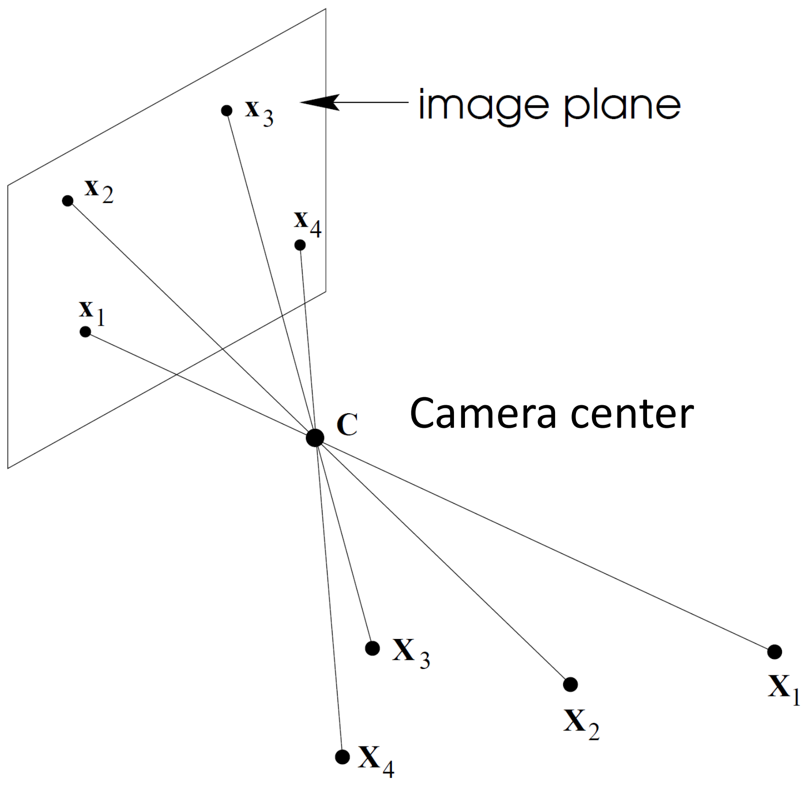



3D World

# Geometry in Image Generation



# 2D Points and 3D Points




- A 2D point is usually used to indicate pixel coordinates of a pixel

$$\mathbf{x} = (x, y) \in \mathcal{R}^2 \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- A 3D point in the real world

$$\mathbf{X} = (x, y, z) \in \mathcal{R}^3 \qquad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

# Homogeneous Coordinates

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = w \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene coordinates

Up to scale

Conversion

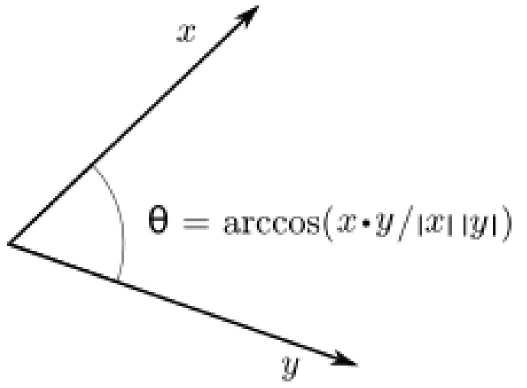
$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

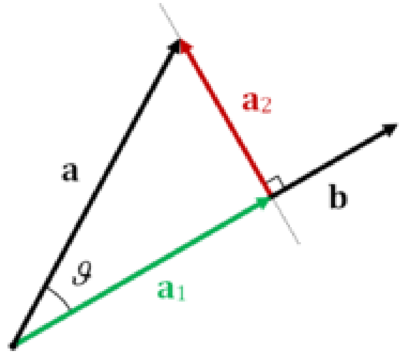
$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

# Vector Inner Product

- Dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$




Vector Projection

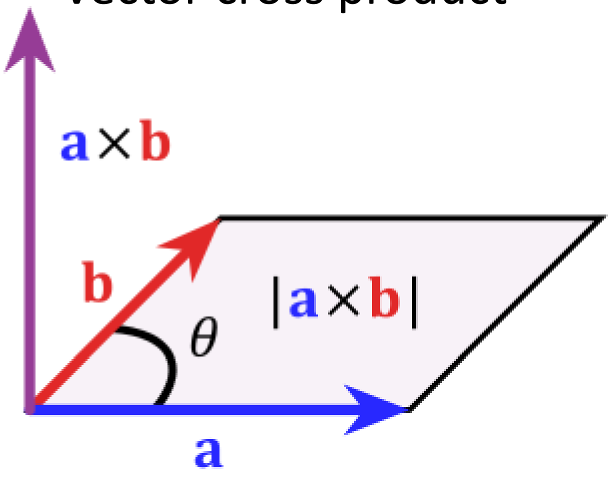
$$a_1 = \|\mathbf{a}\| \cos\theta = \|\mathbf{a}\| \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$

$$\mathbf{a}_1 = a_1 \hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \frac{\mathbf{b}}{\|\mathbf{b}\|}$$

https://en.wikipedia.org/wiki/Dot_product

# Vector Cross Product

Vector cross product




$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \, \mathbf{n}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$
$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

https://en.wikipedia.org/wiki/Cross_product

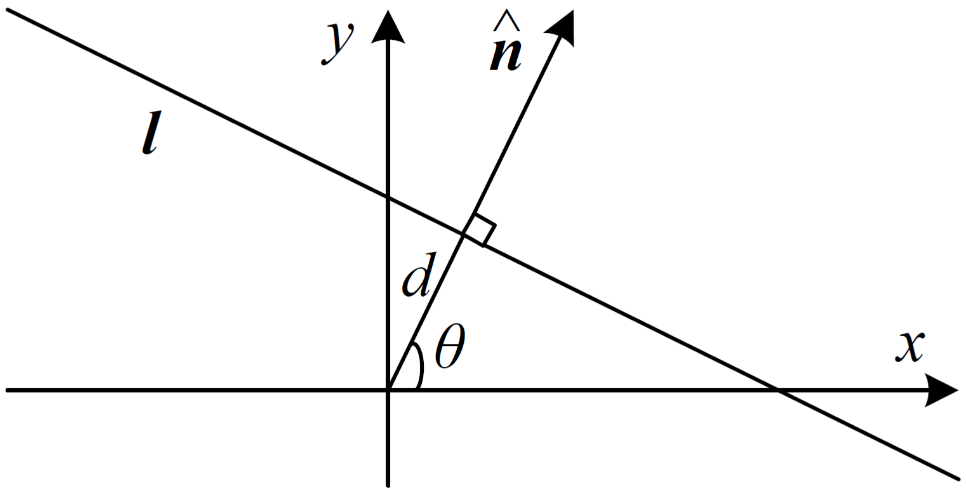
# 2D Lines

- A line in a 2D plane $ax + by + c = 0$ $\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- It is parameterized by $\mathbf{l} = (a, b, c)^T$ Homogeneous Coordinates

  $k(a, b, c)^T$ represents the same line for nonzero k
- Line equation

$$\mathbf{x}^T \mathbf{l} = 0 \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

# 2D Lines




$$\mathbf{l} = (a, b, c)$$

Normalize by $\sqrt{a^2 + b^2}$

$$\mathbf{l} = (\hat{n}_x, \hat{n}_y, d) = (\hat{\mathbf{n}}, d)$$

Normal vector $\|\hat{\mathbf{n}}\| = 1$

Distance to the origin $d$

$$\hat{\mathbf{n}} = (\hat{n}_x, \hat{n}_y) = (\cos\theta, \sin\theta)$$

polar coordinates $(\theta, d)$

# Intersection of 2D Lines

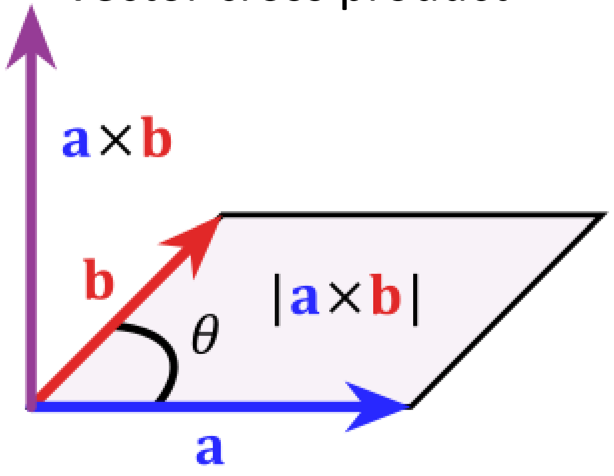
$$\mathbf{l} = (a, b, c)^T \qquad \mathbf{l}' = (a', b', c')^T$$

The intersection is $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

$$\mathbf{l} \cdot (\mathbf{l} \times \mathbf{l}') = \mathbf{l}' \cdot (\mathbf{l} \times \mathbf{l}') = 0$$

$$\mathbf{l}^T \mathbf{x} = \mathbf{l}'^T \mathbf{x} = 0$$

Vector cross product




$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \, \mathbf{n}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Vector dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

A scalar

# A Line Joining two Points

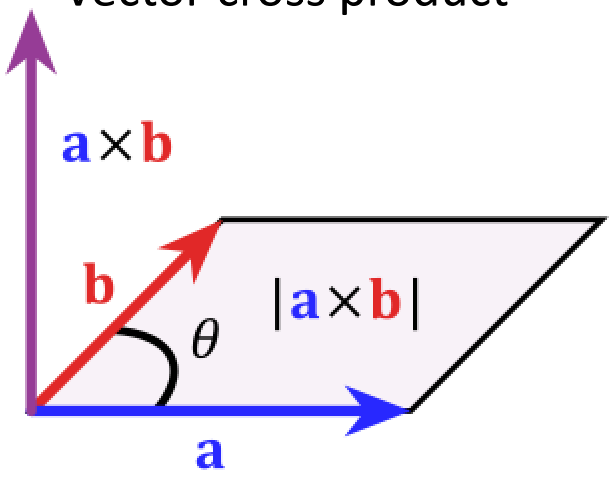
$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$\mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

$$\mathbf{x} \cdot (\mathbf{x} \times \mathbf{x}') = \mathbf{x}' \cdot (\mathbf{x} \times \mathbf{x}') = 0$$

$$\mathbf{x}^T \mathbf{l} = \mathbf{x}'^T \mathbf{l} = 0$$

Vector cross product




$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \, \mathbf{n}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Vector dot product

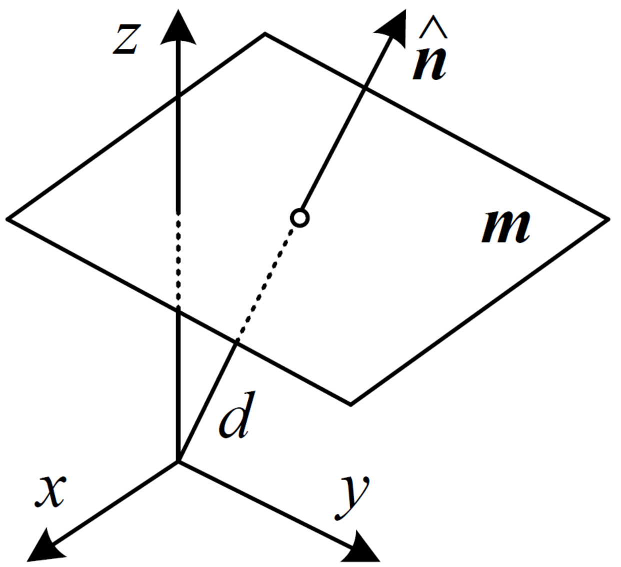
$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

# 3D Plane

- A 3D plane equation $ax + by + cz + d = 0$
- It is parameterized by $(a, b, c, d)$
- Normal vector and distance

$$\mathbf{m} = (\hat{n}_x, \hat{n}_y, \hat{n}_z, d) = (\hat{\mathbf{n}}, d)$$

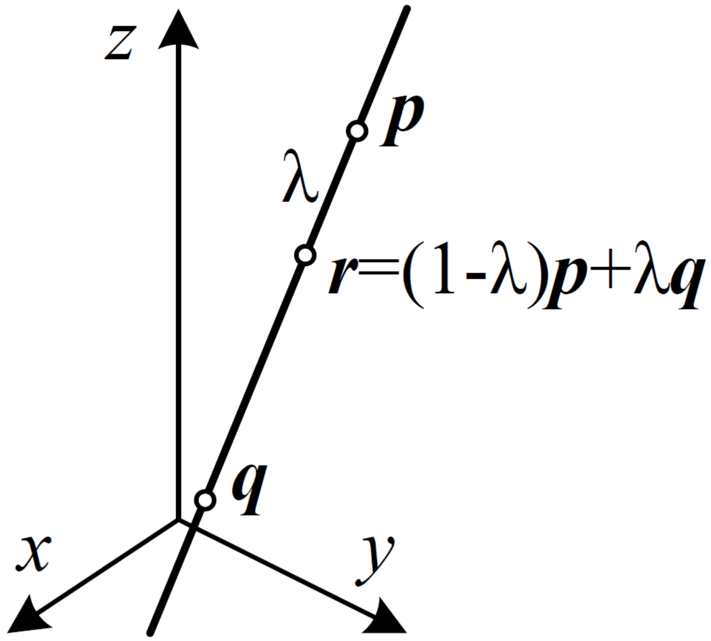
$$\hat{\mathbf{n}} = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$$



# 3D Lines

- Any point on the line is a linear combination of two points

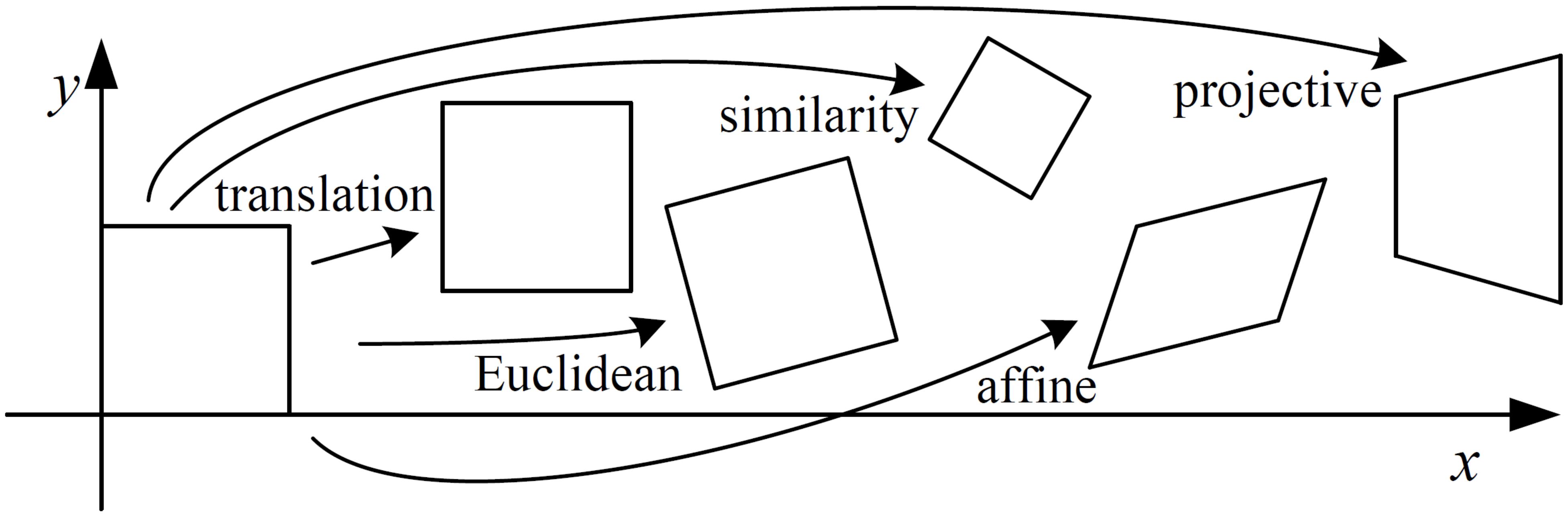
$$\mathbf{r} = (1-\lambda)\mathbf{p} + \lambda\mathbf{q}$$




- Using a line direction

$$\mathbf{r} = \mathbf{p} + \lambda\hat{\mathbf{d}}$$

# 2D Transformations



# 2D Translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

$$2 \times 3$$

augmented vector $\bar{\mathbf{x}} = (x, y, 1)$

Homogeneous coordinate

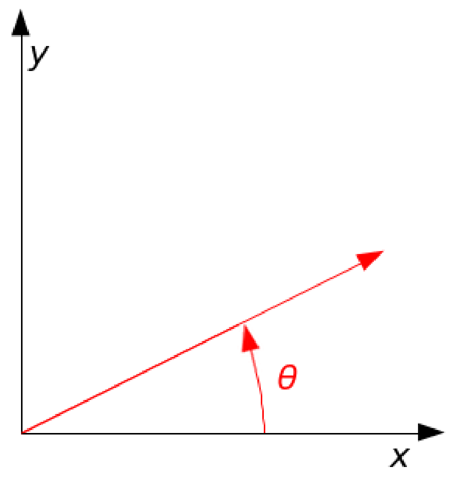
$$\bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \bar{\mathbf{x}}$$

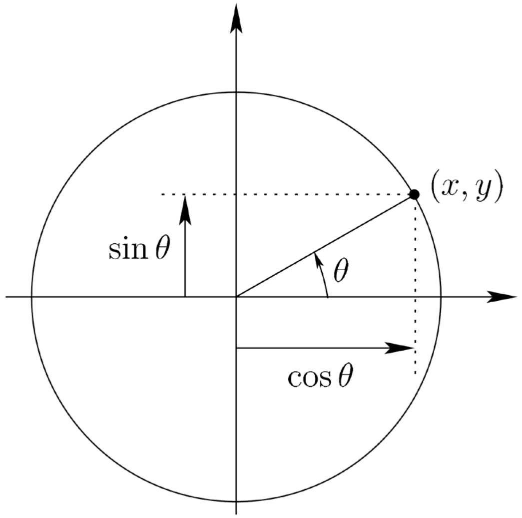
$$3 \times 3$$

# 2D Euclidean Transformation

- 2D Rotation + 2D translation

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$




$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

orthonormal rotation matrix

$$\mathbf{R}\mathbf{R}^T = \mathbf{I} \text{ and } |\mathbf{R}| = 1$$

# 2D Euclidean Transformation

- 2D Rotation + 2D translation

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{x}' = \underset{2 \times 3}{\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}} \bar{\mathbf{x}}$$

$$\bar{\mathbf{x}} = (x, y, 1)$$

- Degree of freedom (DOF)
  - The maximum number of logically independent values
  - 2D Rotation?
  - 2D Euclidean transformation?

# 2D Similarity Transformation

- Scaled 2D rotation + 2D translation

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}} = \begin{bmatrix} a & -b & t_x \\ b & a & t_y \end{bmatrix} \bar{\mathbf{x}} \qquad \bar{\mathbf{x}} = (x, y, 1)$$

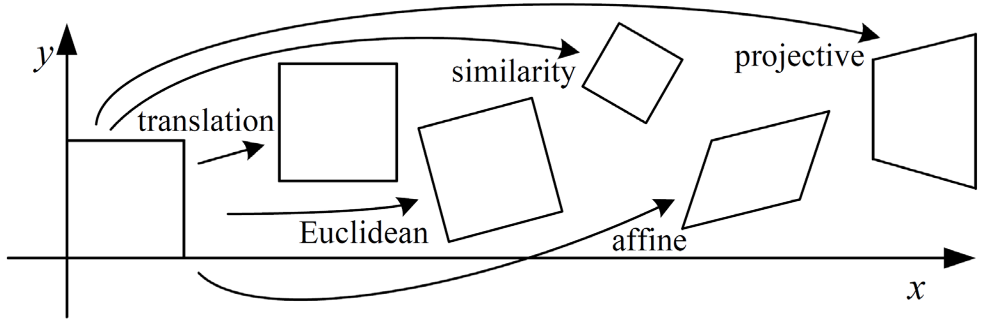
The similarity transform preserves angles between lines.

# 2D Affine Transformation

- Arbitrary 2x3 matrix

$$\mathbf{x}' = \mathbf{A}\bar{\mathbf{x}} \qquad \bar{\mathbf{x}} = (x, y, 1)$$

$$\mathbf{x}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \bar{\mathbf{x}}$$




Parallel lines remain parallel under affine transformations.

# 2D Projective Transformation

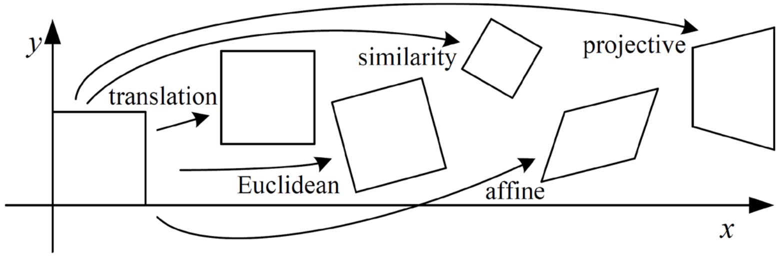
- Also called perspective transform or homography

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}}$$ homogeneous coordinates

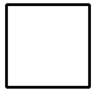
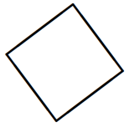
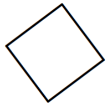
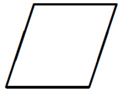
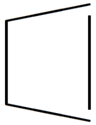
$3 \times 3$ $\tilde{\mathbf{H}}$ is only defined up to a scale

$$x' = \frac{h_{00}x + h_{01}y + h_{02}}{h_{20}x + h_{21}y + h_{22}} \quad \text{and} \quad y' = \frac{h_{10}x + h_{11}y + h_{12}}{h_{20}x + h_{21}y + h_{22}}$$

Perspective transformations preserve straight lines



# Hierarchy of 2D Transformations

| Transformation | Matrix | # DoF | Preserves | Icon |
|---|---|---|---|---|
| translation | $\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{2\times 3}$ | 2 | orientation |  |
| rigid (Euclidean) | $\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{2\times 3}$ | 3 | lengths |  |
| similarity | $\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{2\times 3}$ | 4 | angles |  |
| affine | $\begin{bmatrix} \mathbf{A} \end{bmatrix}_{2\times 3}$ | 6 | parallelism |  |
| projective | $\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{3\times 3}$ | 8 | straight lines |  |

# 3D Translation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

$$3 \times 4$$

augmented vector $\bar{\mathbf{x}} = (x, y, z, 1)$

# 3D Euclidean Transformation SE(3)

- 3D Rotation + 3D translation

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

$$3 \times 4$$

$$\bar{\mathbf{x}} = (x, y, z, 1)$$

orthonormal rotation matrix

$$\mathbf{R}\mathbf{R}^T = \mathbf{I} \text{ and } |\mathbf{R}| = 1$$

$$3 \times 3$$

We will focus on 3D rotations in next lecture.

# 3D Similarity Transformation

- Scaled 3D rotation + 3D translation

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \underbrace{\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}}_{3 \times 4} \bar{\mathbf{x}} \qquad \bar{\mathbf{x}} = (x, y, z, 1)$$

This transformation preserves angles between lines and planes.

# 3D Affine Transformation

$$\mathbf{x}' = \mathbf{A}\bar{\mathbf{x}} \qquad \bar{\mathbf{x}} = (x, y, z, 1)$$

$$\mathbf{x}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \end{bmatrix} \bar{\mathbf{x}}$$

$$3 \times 4$$

Parallel lines and planes remain parallel under affine transformations.

# 3D Projective Transformation

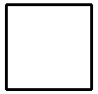
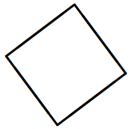
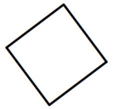
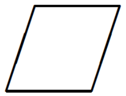
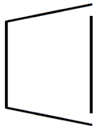
- Also called 3D perspective transform or homography

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}}$$ homogeneous coordinates

$4 \times 4$ $\tilde{\mathbf{H}}$ is only defined up to a scale

- Perspective transformations preserve straight lines

# 3D Transformations

| Transformation | Matrix | # DoF | Preserves | Icon |
|---|---|---|---|---|
| translation | $\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{3\times 4}$ | 3 | orientation |  |
| rigid (Euclidean) | $\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$ | 6 | lengths |  |
| similarity | $\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$ | 7 | angles |  |
| affine | $\begin{bmatrix} \mathbf{A} \end{bmatrix}_{3\times 4}$ | 12 | parallelism |  |
| projective | $\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{4\times 4}$ | 15 | straight lines |  |

# Further Reading

- Section 2.1, Computer Vision, Richard Szeliski

- Chapter 2 and 3, Multiple View Geometry in Computer Vision, Richard Hartley and Andrew Zisserman