Epipolar Geometry and Stereo

CS 6384 Computer Vision
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Depth Perception

• Metric
  • The car is 10 meters away

• Ordinary
  • The tree is behind the car
Depth Cues

• Information for sensory stimulation that is relevant to depth perception

• Monocular cues: single eye

• Stereo cues: both eyes

“Paris Street, Rainy Day,” Gustave Caillebotte, 1877. Art Institute of Chicago

• Texture of the bricks
• Perspective projection
• Etc.
Monocular Depth Cues

- Retina image size
- Height in visual field
- Shadow
- Occlusions
- Image blur
- Motion parallax (relative difference in speed) Further objects move slower
- Atmospheric cue further away because it has lower contrast
Stereo Depth Cues

• Binocular disparity

• Each eye provides a different viewpoint, which results in different images on the retina
Epipolar Geometry

• The geometry of stereo vision
  • Given 2D images of two views

  • What is the relationship between pixels of the images?

  • Can we recover the 3D structure of the world from the 2D images?
Epipolar Geometry

3D point

epipolar plane \( \pi \)

2D projection 1

2D projection 2

Camera center 1

Camera center 2

\( R, t \)
Epipolar Geometry
Epipolar Geometry

Rotation and Translation between two views
Epipolar Geometry

• What is the mapping for a point in one image to its epipolar line?

\[ x \mapsto l' \]
Fundamental Matrix

- Recall camera projection
  \[ P = K[R|t] \]
  \[ \mathbf{x} = P\mathbf{X} \]
  Homogeneous coordinates

- Backprojection
  \[ P^+ \mathbf{x} \] and \( C \) are two points on the ray
  \[ P^+ \] is the pseudo-inverse of \( P \), \( PP^+ = I \)
  \[ P^+ = P^T(PP^T)^{-1} \]
  \[ \mathbf{X}(\lambda) = (1 - \lambda)P^+\mathbf{x} + \lambda C \]
Fundamental Matrix

- Project to the other image
  - \( P^+x \) and \( C \) are two points on the ray
  - Epipolar line
    - \( l' = (P'C) \times (P'P^+x) \)
    - Epipole \( e' = (P'C) \)
    - \( l' = [e'] \times (P'P^+x) \)

Cross product matrix
Fundamental Matrix

- Epipolar line

\[ l' = [e'] \times (P' P^+ x) = Fx \]

- Fundamental matrix

\[ F = [e'] \times P' P^+ \]

3x3

\[ l' = Fx \]
Fundamental Matrix

\[
F = [\mathbf{e'}] \times P' P^+ \quad \mathbf{e'} = (P' \mathbf{C})
\]

\[
P = K[I | 0] \quad P' = K'[R | t]
\]

\[
P^+ = \begin{bmatrix}
K^{-1} \\
0^T
\end{bmatrix}
\]

\[
\mathbf{C} = \begin{pmatrix}
0 \\
1
\end{pmatrix}
\]

\[
e = P \left( \begin{pmatrix} -R^T t \\ 1 \end{pmatrix} \right) = KR^t t \\
e' = P' \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = K't
\]

\[
F = [\mathbf{e'}] \times K'R K^{-1} = K'^{-T} [t] \times RK^{-1} = K'^{-T} R [R^T t] \times K^{-1} = K'^{-T} RK^T [KR t] \times
\]
Properties of Fundamental Matrix

\[ x' \text{ is on the epipolar line } \quad l' = Fx \]
\[ x'^T F x = 0 \]

- Transpose: if F is the fundamental matrix of \((P, P')\), then \(F^T\) is the fundamental matrix of \((P', P)\)
- Epipolar line: \( l' = Fx \quad l = F^T x' \)
- Epipoles:
  \[ e'^T F = 0 \quad Fe = 0 \]
  \[ e'^T (Fx) = (e'^T F)x = 0 \text{ for all } x \]
- 7 degrees of freedom
  \[ \det F = 0 \]
Why the Fundamental Matrix is Useful?

\[ l' = Fp \]
Estimating the Fundamental Matrix

• The 8-point algorithm
Estimating the Fundamental Matrix

\[ x'^T F x = 0 \quad x = (x, y, 1)^T \quad x' = (x', y', 1)^T \]

\[
x' x f_{11} + x' y f_{12} + x' f_{13} + y' x f_{21} + y' y f_{22} + y' f_{23} + x f_{31} + y f_{32} + f_{33} = 0
\]

\[
(x' x, x' y, x', y' x, y' y, y', x, y, 1)^T f = 0
\]

n correspondences

\[
A f = \begin{bmatrix}
  x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  x'_n x_n & x'_n y_n & x'_n & y'_n x_n & y'_n y_n & y'_n & x_n & y_n & 1
\end{bmatrix} f = 0
\]

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Linear System

\[ A f = 0 \]
\[ n \times 9 \quad 9 \times 1 \]

- Find non-zero solutions
- If \( f \) is a solution, \( k \times f \) is also a solution for \( k \in \mathbb{R} \)
- If the rank of \( A \) is 8, unique solution (up to scale)
- Otherwise, we can seek a solution \( ||f|| = 1 \)

\[ \min ||Af|| \quad \text{Subject to} \quad ||f|| = 1 \]

Solution: \[ A = U D V^T \]

\[ n \times 9 \quad 9 \times 9 \quad 9 \times 9 \]

\( f \) is the last column of \( V \)

A5.3 in HZ
Estimating the Fundamental Matrix

• The singularity constraint \[ \det F = 0 \]

\[
\min \|F - F'\| \\
\text{Subject to } \det F' = 0
\]

\[ F = UDV^T \]

\[ D = \text{diag}(r, s, t) \]

\[ r \geq s \geq t \]

Solution:

\[ F' = U\text{diag}(r, s, 0)V^T \]
Special Case: A Stereo System

- The right camera is shifted by $T_x$ (the stereo baseline)
- The camera intrinsics are the same
Special Case: A Stereo System

- Left camera
  \[ x_l = f \frac{X}{Z} + p_x \quad y_l = f \frac{Y}{Z} + p_y \]

- Right camera
  \[ x_r = f \frac{X - T_x}{Z} + p_x \]
  \[ y_r = f \frac{Y}{Z} + p_y \]
Stereo Disparity

- Disparity
  \[ d = x_l - x_r \]
  \[ = \left( f \frac{X}{Z} + p_x \right) - \left( f \frac{X - T_x}{Z} + p_x \right) \]
  \[ = f \frac{T_x}{Z} \]

- Depth
  \[ Z = f \frac{T_x}{d} \]

Recall motion parallax: near objects move faster (large disparity)
Special Case: A Stereo System

\[ P = K[I \mid 0] \quad P' = K[I \mid t] \]

\[ F = [e']_x K'RK^{-1} = K^{-T}[t]_x R K^{-1} = K'^{-T}R[R^T t]_x K^{-1} = K'^{-T}RK^T[e]_x \]

\[ F = [e']_x KK^{-1} = [e']_x \]

\[ e' = (P'C) \]

\[ C = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -f_x T_x \\ 0 & f_x T_x & 0 \end{bmatrix} \]

\[ x^T F x = 0 \]

\[ y = y' \]
Stereo Example

Disparity values (0-64)

\[ d = f \frac{T_x}{Z} \]

Note how disparity is larger (brighter) for closer surfaces.
Computing Disparity

- Eipipolar lines are horizontal lines in stereo
- For general cases, we can find correspondences on eipipolar lines
- Depth from disparity

$Z = f \frac{T_x}{d}$
Triangulation

• Compute the 3D point given image correspondences

Intersection of two backprojected lines

\[ X = l \times l' \]
Triangulation

• In practice, we find the correspondences \( y \quad y' \)

• The backprojected lines may not intersect

• Find \( X^* \) that minimizes

\[
d(y, PX^*) + d(y', P'X^*)
\]

Projection matrix
Further Reading

• Stanford CS231A: Computer Vision, From 3D Reconstruction to Recognition, Lecture 5 [https://web.stanford.edu/class/cs231a/syllabus.html](https://web.stanford.edu/class/cs231a/syllabus.html)

• Multiview Geometry in Computer Vision, Richard Hartley and Andrew Zisserman, Chapter 9, Epipolar Geometry and Fundamental Matrix