




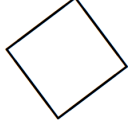
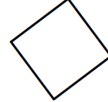
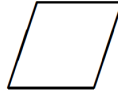
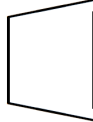
3D Rotations

CS 6384 Computer Vision

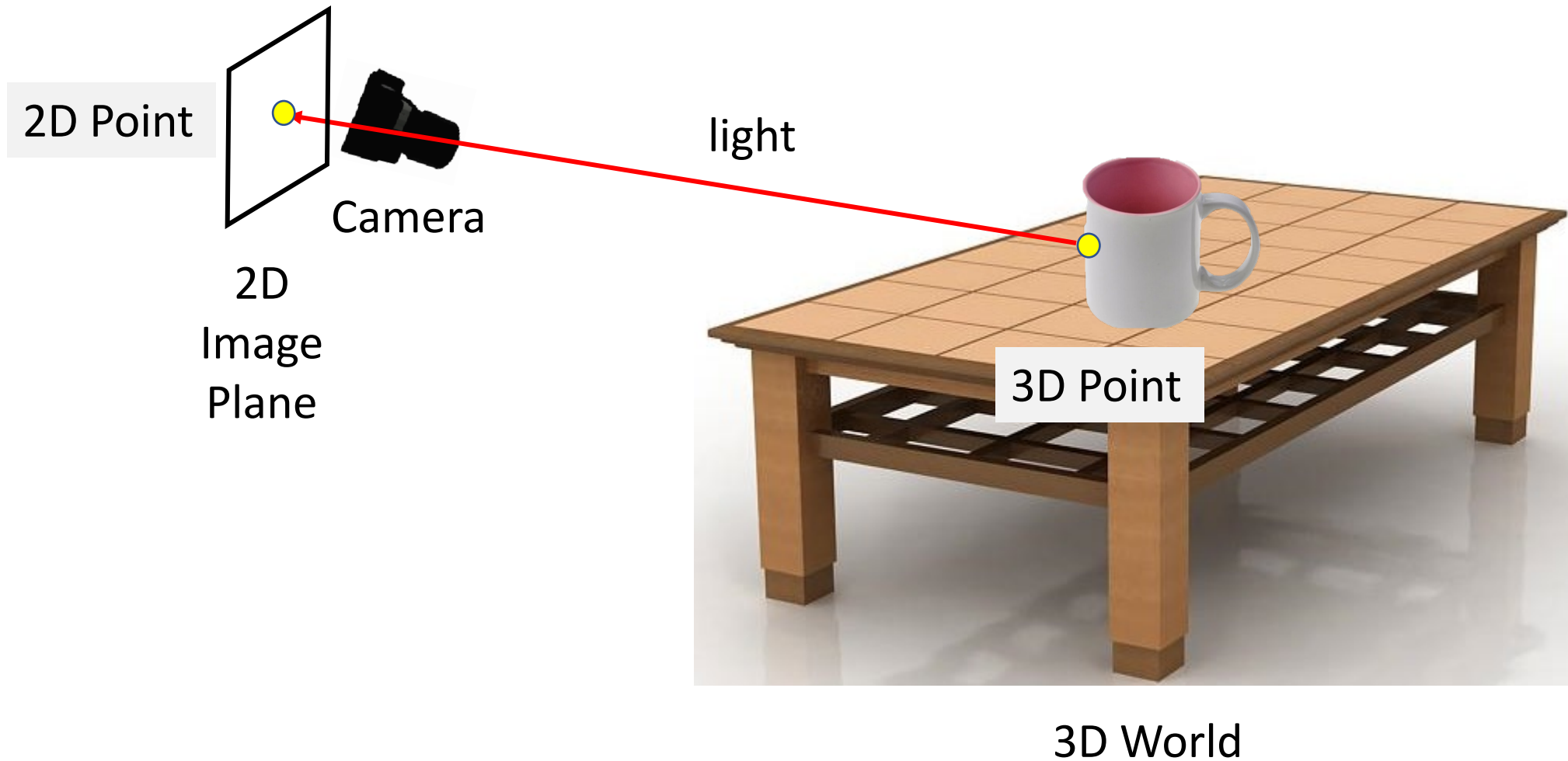
Professor Yu Xiang

The University of Texas at Dallas

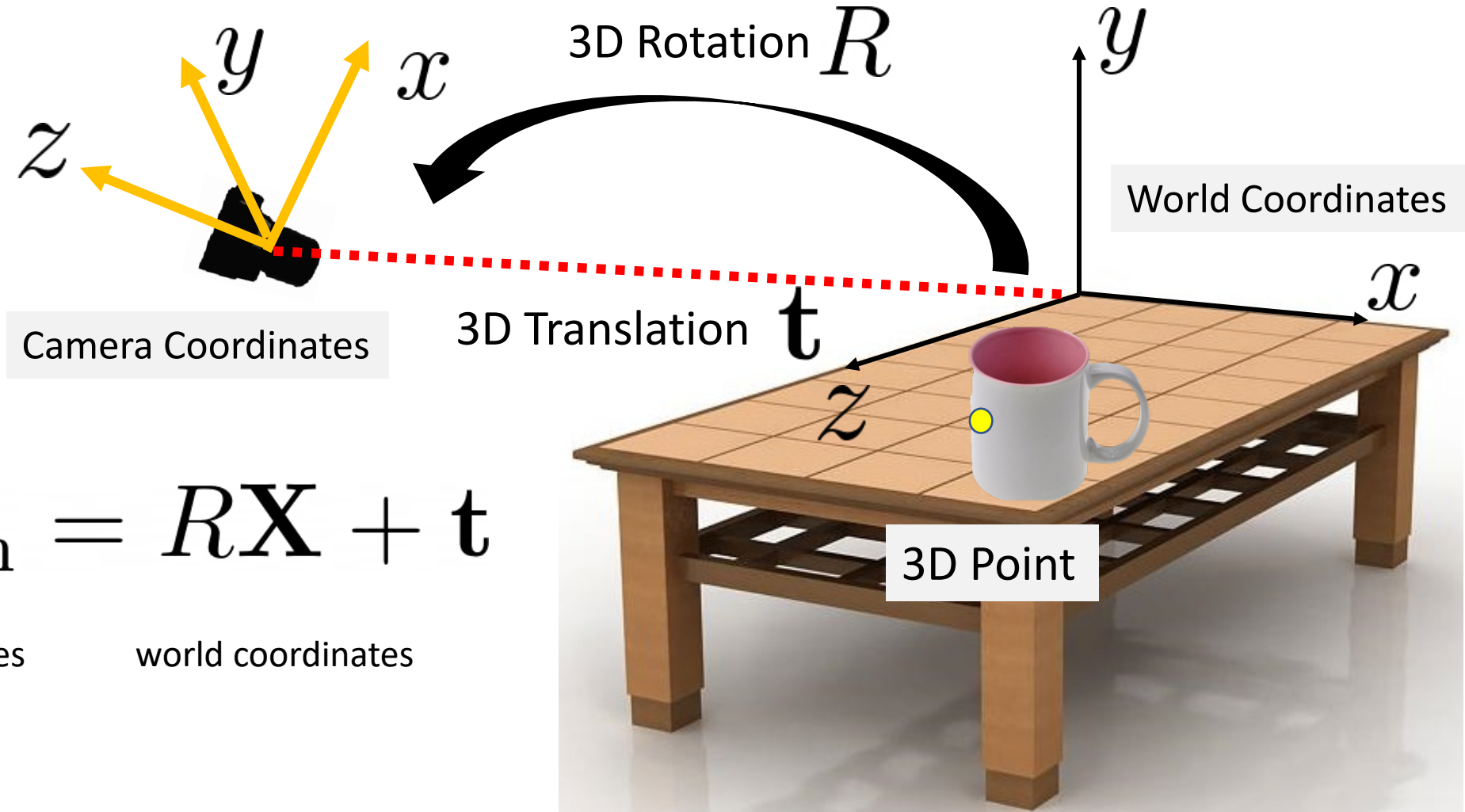
Recall 3D Transformations

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{3 \times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{3 \times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{3 \times 4}$	7	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{3 \times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{4 \times 4}$	15	straight lines	

Recall Geometry in Image Generation



Camera Rotation and Translation



$$\mathbf{X}_{\text{cam}} = R\mathbf{X} + \mathbf{t}$$

camera coordinates

world coordinates

An Example for Camera Pose Tracking

ORB-SLAM



<https://webdiis.unizar.es/~raulmur/orbslam/>

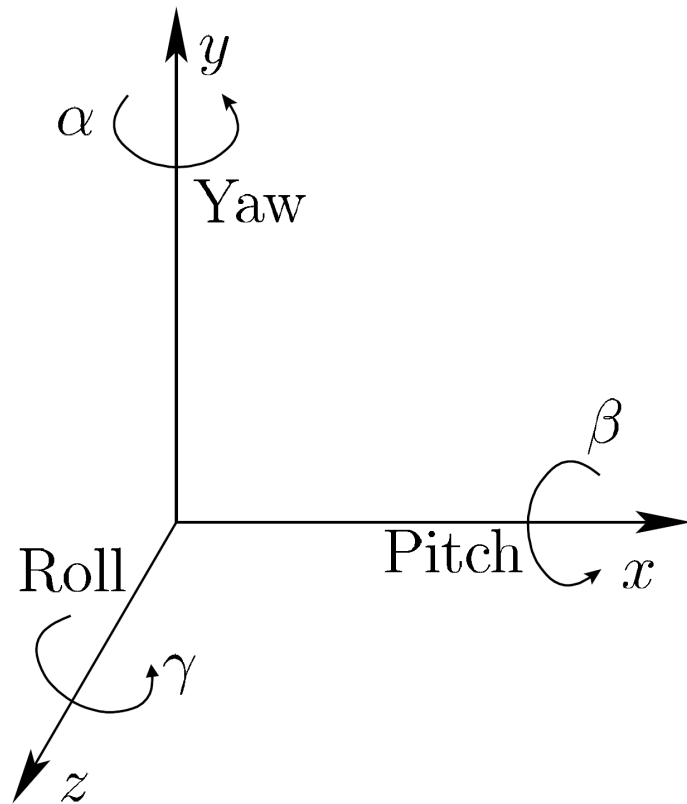
3D Rotations

- Unit-length columns
- Perpendicular columns
- $\det M = 1$
- 3 DOFs

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Euler Angles: Yaw, Pitch, Roll

- Counterclockwise rotation



Roll $R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Pitch $R_x(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{bmatrix}$

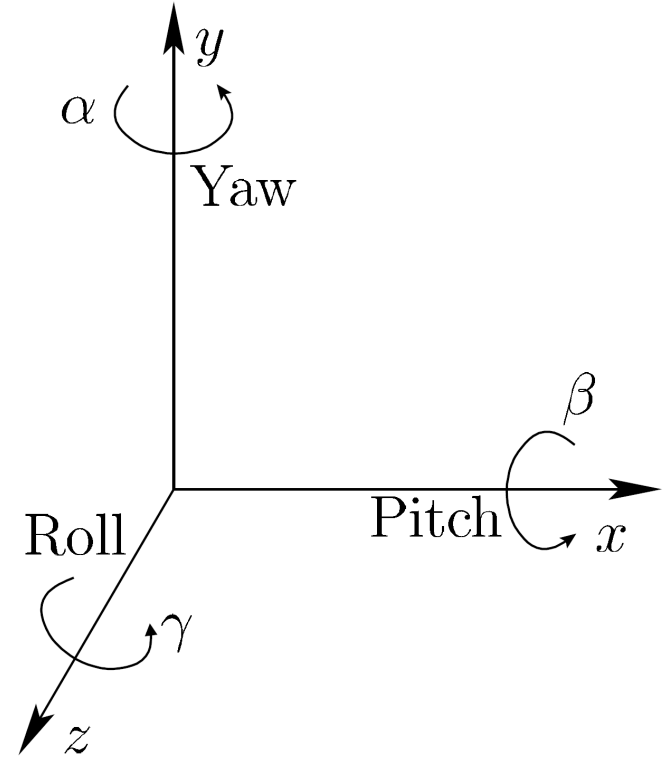
Yaw $R_y(\alpha) = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}$

Combining Rotations

- Matrix multiplications are “backwards”

$$R(\alpha, \beta, \gamma) = R_y(\alpha)R_x(\beta)R_z(\gamma)$$

$$\alpha, \gamma \in [0, 2\pi] \quad \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

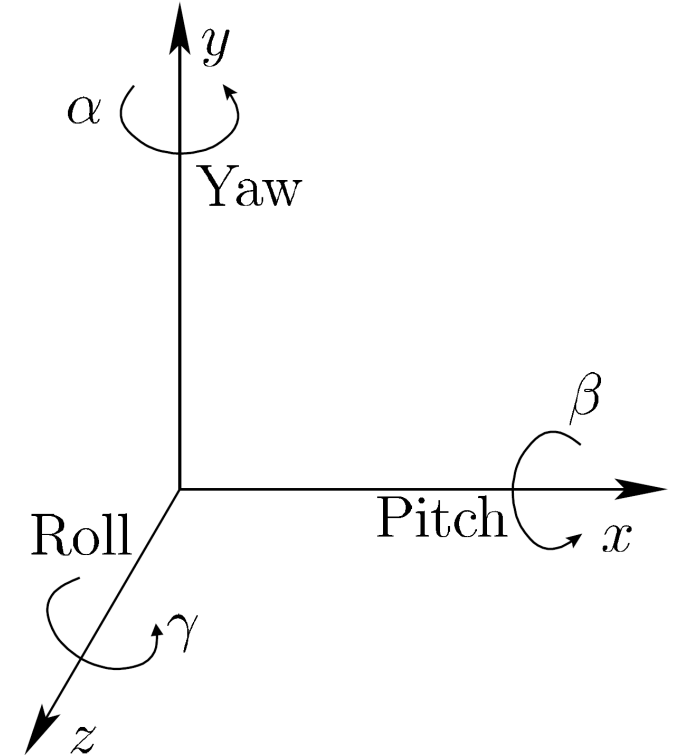


The Order Matters

- 12 possible sequences of rotation axes

Proper Euler angles ($z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y$)

Tait–Bryan angles ($x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z$)

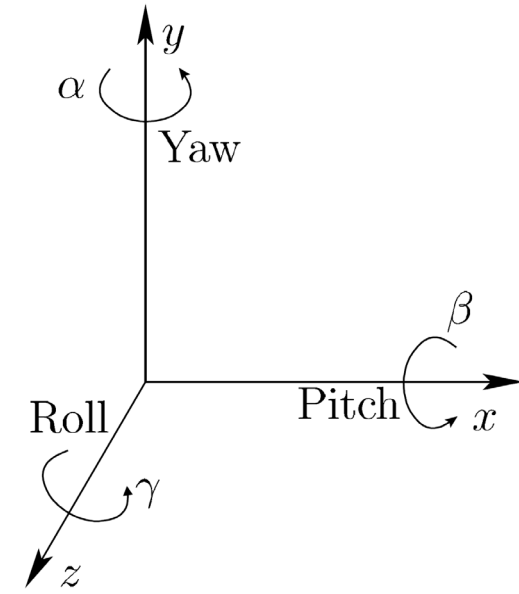


Kinematic Singularities

- When pitch $\beta = \frac{\pi}{2}$

$$R(\alpha, \beta, \gamma) = R_y(\alpha)R_x(\beta)R_z(\gamma)$$

$$\begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha - \gamma) & \sin(\alpha - \gamma) & 0 \\ 0 & 0 & -1 \\ -\sin(\alpha - \gamma) & \cos(\alpha - \gamma) & 0 \end{bmatrix}$$



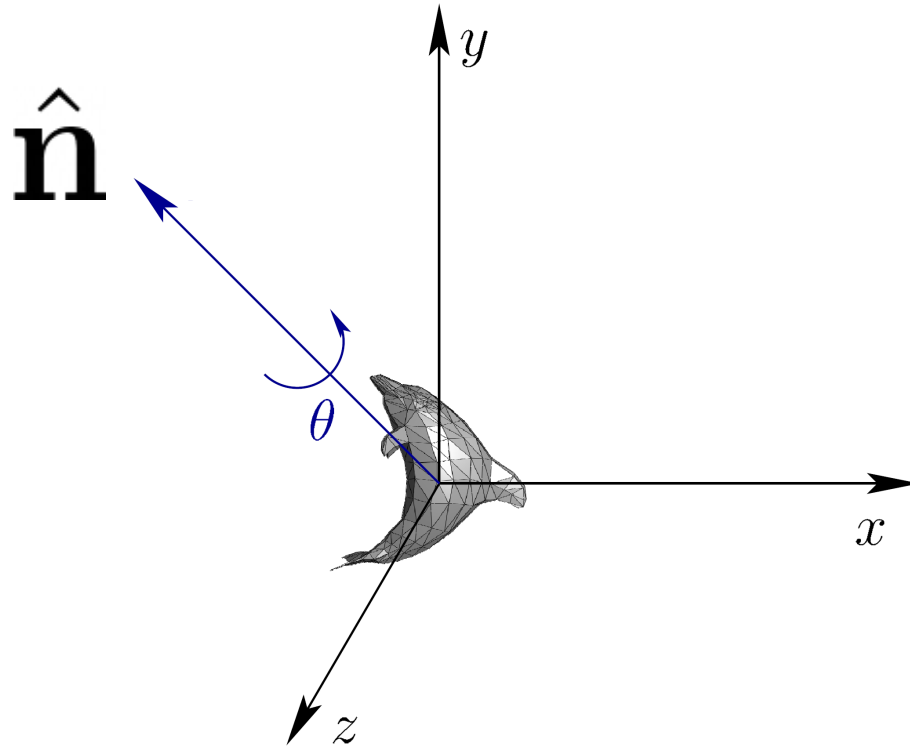
Longitude
Latitude

Only one DOF

The kinematic singularity often causes the viewpoint to spin uncontrollably in VR.

Axis-Angle Representations of Rotation

- Euler's rotation theorem: every 3D rotation can be considered as a rotation by an angle about an axis through the origin



Unit vector

$$\hat{\mathbf{n}} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)$$

$$\boldsymbol{\omega} = \theta \hat{\mathbf{n}}$$

DOFs?

Rodrigues' Rotation Formula

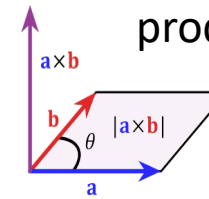
Axis-angle to rotation matrix $\boldsymbol{\omega} = \theta \hat{\mathbf{n}}$

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} + (1 - \cos \theta) [\hat{\mathbf{n}}]_{\times}^2$$

Cross product matrix $[\hat{\mathbf{n}}]_{\times} = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix}$

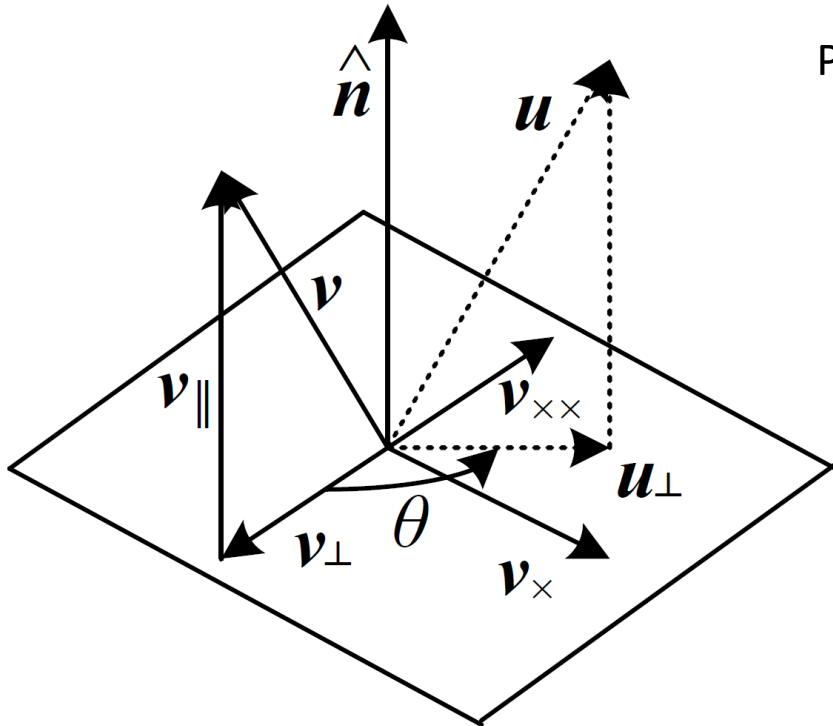
Rodrigues' Rotation Formula

Vector cross product



Vector dot product
 $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$

Rotate \mathbf{v} around $\hat{\mathbf{n}}$ to get \mathbf{u}



$$\boldsymbol{\omega} = \theta \hat{\mathbf{n}}$$

Projection

$$\mathbf{v}_{\parallel} = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{v}) = (\hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{v}$$

Perpendicular residual

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{v}$$

Cross product
(rotation 90°)

$$\mathbf{v}_{\times} = \hat{\mathbf{n}} \times \mathbf{v}_{\perp} = \hat{\mathbf{n}} \times \mathbf{v} = [\hat{\mathbf{n}}]_{\times} \mathbf{v}$$

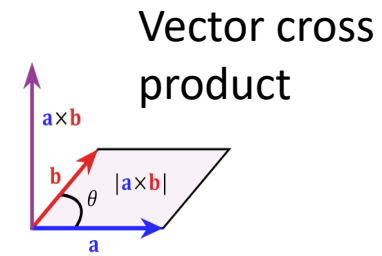
Cross product matrix

$$[\hat{\mathbf{n}}]_{\times} = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix}$$

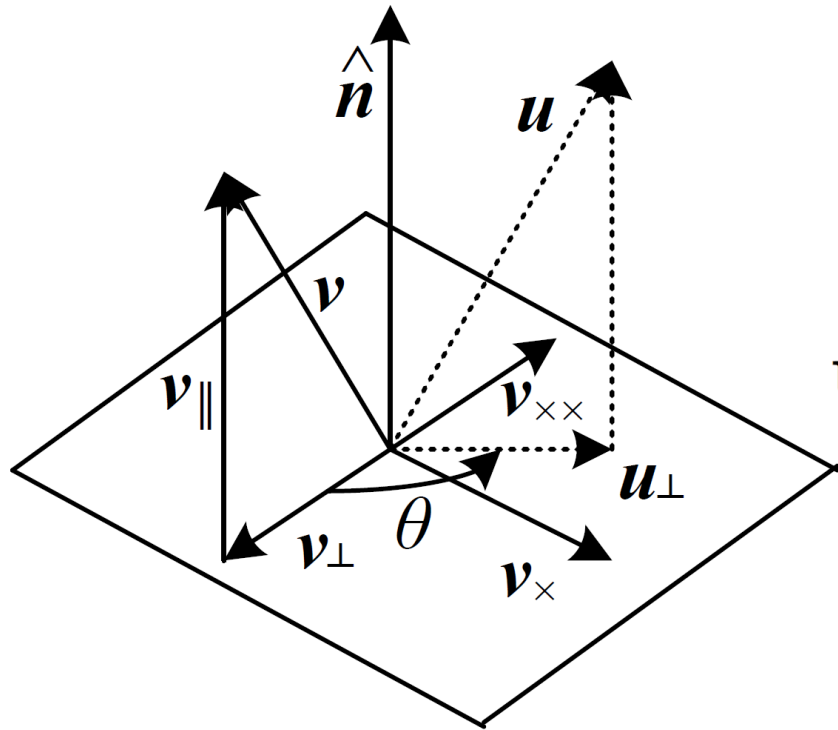
Rotation 90° again $\mathbf{v}_{\times\times} = \hat{\mathbf{n}} \times \mathbf{v}_{\times} = [\hat{\mathbf{n}}]_{\times}^2 \mathbf{v} = -\mathbf{v}_{\perp}$

$$\mathbf{v}_{\parallel} = \mathbf{v} - \mathbf{v}_{\perp} = \mathbf{v} + \mathbf{v}_{\times\times} = (\mathbf{I} + [\hat{\mathbf{n}}]_{\times}^2)\mathbf{v}$$

Rodrigues' Rotation Formula



Vector dot product
 $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$



$$\boldsymbol{\omega} = \theta \hat{\mathbf{n}}$$

In-plane component

$$\mathbf{u}_{\perp} = \cos \theta \mathbf{v}_{\perp} + \sin \theta \mathbf{v}_{\times} = (\sin \theta [\hat{\mathbf{n}}]_{\times} - \cos \theta [\hat{\mathbf{n}}]_{\times}^2) \mathbf{v}$$

$$\mathbf{v}_{\parallel} = \mathbf{v} - \mathbf{v}_{\perp} = \mathbf{v} + \mathbf{v}_{\times \times} = (\mathbf{I} + [\hat{\mathbf{n}}]_{\times}^2) \mathbf{v}$$

$$\mathbf{u} = \mathbf{u}_{\perp} + \mathbf{v}_{\parallel} = (\mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} + (1 - \cos \theta) [\hat{\mathbf{n}}]_{\times}^2) \mathbf{v}$$

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} + (1 - \cos \theta) [\hat{\mathbf{n}}]_{\times}^2$$

$$[\hat{\mathbf{n}}]_{\times} = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix}$$

Rodrigues' formula

Rodrigues' Rotation Formula

$$\boldsymbol{\omega} = \theta \hat{\mathbf{n}} = (\omega_x, \omega_y, \omega_z)$$

Axis-angle

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} + (1 - \cos \theta) [\hat{\mathbf{n}}]_{\times}^2$$

Rotation matrix

For small rotation angles $\sin \theta \approx \theta$, $\cos \theta \approx 1 - \frac{\theta^2}{2} \approx 1$.

$$\mathbf{R}(\boldsymbol{\omega}) \approx \mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} \approx \mathbf{I} + [\theta \hat{\mathbf{n}}]_{\times} = \begin{bmatrix} 1 & -\omega_z & \omega_y \\ \omega_z & 1 & -\omega_x \\ -\omega_y & \omega_x & 1 \end{bmatrix} \quad \text{linearized relationship}$$

$$\mathbf{R}(\boldsymbol{\omega}) \mathbf{v} \approx \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} \quad \text{Derivative} \quad \frac{\partial \mathbf{R} \mathbf{v}}{\partial \boldsymbol{\omega}^T} = -[\mathbf{v}]_{\times} = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}$$

$$\text{Cross product} \quad \boldsymbol{w} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{w}$$

SO(n): Special Orthogonal Group

- SO(n): Space of rotation matrices in \mathbb{R}^n

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : RR^T = I, \det(R) = 1\}$$

- SO(3): space of 3D rotation matrices
- Group is a set G , with an operation \bullet , satisfying the following axioms:
 - Closure: $a \in G, b \in G \Rightarrow a \cdot b \in G$
 - Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c), \forall a, b, c \in G$
 - Identity element: $\exists e \in G, e \cdot a = a, \forall a \in G$
 - Inverse element: $\forall a \in G, \exists b \in G, a \cdot b = b \cdot a = e$

Exponential Map for SO(3)

- Matrix exponential $\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$ $X : n \times n$
factorial

Rodrigues' formula

$$\boldsymbol{\omega} = \theta \hat{\mathbf{n}} \quad \mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} + (1 - \cos \theta) [\hat{\mathbf{n}}]_{\times}^2$$

$$\exp[\boldsymbol{\omega}]_{\times} = \mathbf{I} + \theta [\hat{\mathbf{n}}]_{\times} + \frac{\theta^2}{2} [\hat{\mathbf{n}}]_{\times}^2 + \frac{\theta^3}{3!} [\hat{\mathbf{n}}]_{\times}^3 + \dots$$

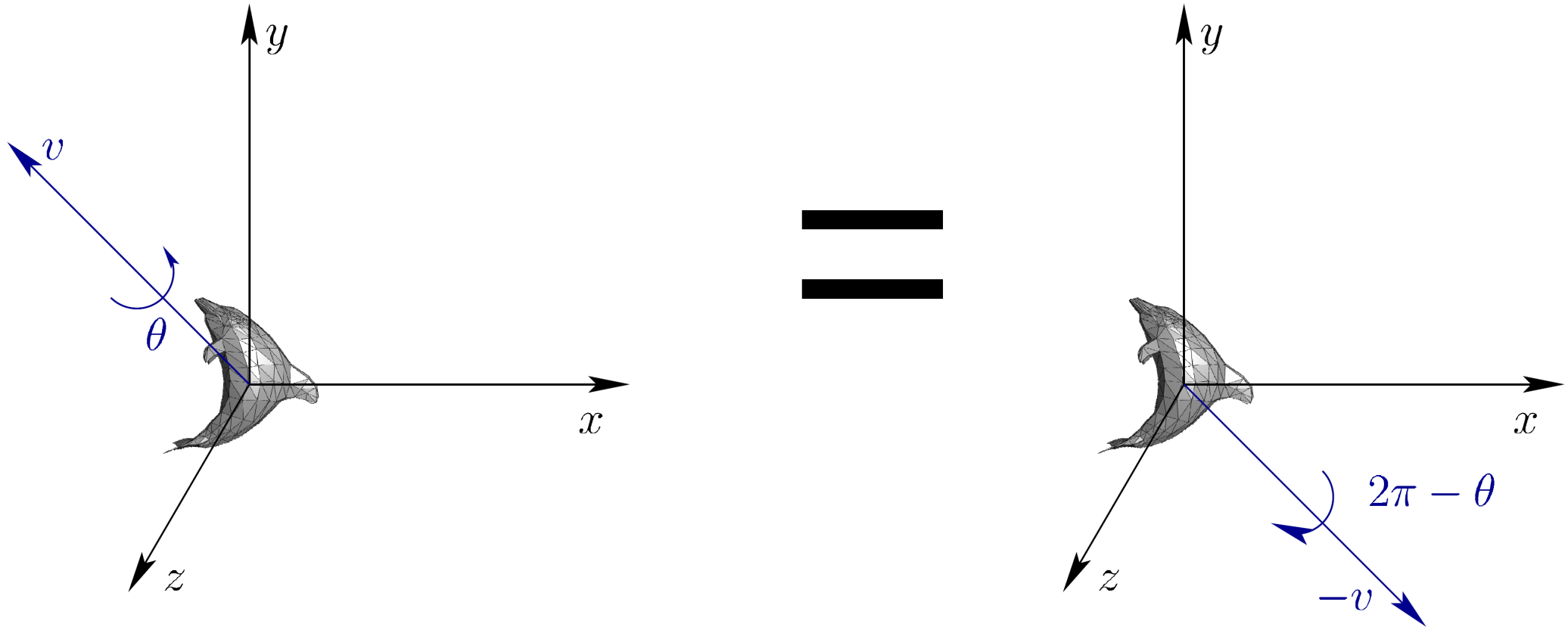
$$= \mathbf{I} + \left(\theta - \frac{\theta^3}{3!} + \dots\right) [\hat{\mathbf{n}}]_{\times} + \left(\frac{\theta^2}{2} - \frac{\theta^4}{4!} + \dots\right) [\hat{\mathbf{n}}]_{\times}^2$$

$$= \mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} + (1 - \cos \theta) [\hat{\mathbf{n}}]_{\times}^2$$

$$[\hat{\mathbf{n}}]_{\times}^{k+2} = -[\hat{\mathbf{n}}]_{\times}^k$$

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \exp[\boldsymbol{\omega}]_{\times}$$

Two-to-one Problem of Axis-Angle Representations



Quaternions

- Quaternions generalize complex numbers and can be used to represents 3D rotations

$$q = w + \underbrace{xi + yj + zk}$$

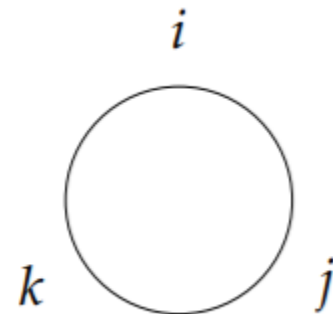
↑
Scale (real part) Vector (imaginary part)

- Properties $i^2 = j^2 = k^2 = -1$

$$ij = k, ji = -k$$

$$jk = i, kj = -i$$

$$ki = j, ik = -j$$



Quaternion Addition and Multiplication

- Addition

$$p + q = (p_0 + q_0) + (p_1 + q_1)\mathbf{i} + (p_2 + q_2)\mathbf{j} + (p_3 + q_3)\mathbf{k}$$

- Multiplication

$$\begin{aligned} pq &= (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) \\ &= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + p_0(q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) + q_0(p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) \\ &\quad + (p_2q_3 - p_3q_2)\mathbf{i} + (p_3q_1 - p_1q_3)\mathbf{j} + (p_1q_2 - p_2q_1)\mathbf{k}. \end{aligned}$$

$$pq = p_0q_0 - \mathbf{p} \cdot \mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}$$

$$\mathbf{p} = (p_1, p_2, p_3) \quad \mathbf{q} = (q_1, q_2, q_3)$$

Complex Conjugate, Norm and Inverse

- Conjugate $q = q_0 + \mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$
 $q^* = q_0 - \mathbf{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$

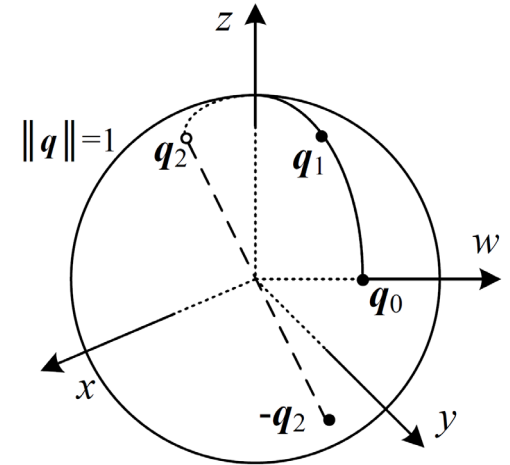
- Norm $|q| = \sqrt{q^*q}$
 $q^*q = (q_0 - \mathbf{q})(q_0 + \mathbf{q})$
 $= q_0q_0 - (-\mathbf{q}) \cdot \mathbf{q} + q_0\mathbf{q} + (-\mathbf{q})q_0 + (-\mathbf{q}) \times \mathbf{q}$
 $= q_0^2 + \mathbf{q} \cdot \mathbf{q}$
 $= q_0^2 + q_1^2 + q_2^2 + q_3^2$
 $= qq^*.$

- Inverse $q^{-1} = \frac{q^*}{|q|^2}$ $q^{-1}q = qq^{-1} = 1$

Unit Quaternions as 3D Rotations

- For unit quaternions, axis-angle

$$q = (w, \mathbf{v}) = \left(\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \hat{\mathbf{n}} \right)$$



- For $\mathbf{v} \in \mathbb{R}^3$, rotation according to a unit quaternion $q = q_0 + \mathbf{q}$

$$\begin{aligned} L_q(\mathbf{v}) &= \mathbf{q}\mathbf{v}\mathbf{q}^* \\ &= (q_0^2 - \|\mathbf{q}\|^2)\mathbf{v} + 2(\mathbf{q} \cdot \mathbf{v})\mathbf{q} + 2q_0(\mathbf{q} \times \mathbf{v}) \end{aligned}$$

The real part of \mathbf{v} is 0

Unit Quaternions as 3D Rotations

$$q = (w, \mathbf{v}) = \left(\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \hat{\mathbf{n}} \right)$$

$$\begin{aligned} \mathbf{R}(\hat{\mathbf{n}}, \theta) &= \mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} + (1 - \cos \theta) [\hat{\mathbf{n}}]_{\times}^2 \\ &= \mathbf{I} + 2w [\mathbf{v}]_{\times} + 2[\mathbf{v}]_{\times}^2. \end{aligned}$$

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$(1 - \cos \theta) = 2 \sin^2 \frac{\theta}{2}$$

$$[\mathbf{v}]_{\times} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

$$[\mathbf{v}]_{\times}^2 = \begin{bmatrix} -y^2 - z^2 & xy & xz \\ xy & -x^2 - z^2 & yz \\ xz & yz & -x^2 - y^2 \end{bmatrix}$$

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - zw) & 2(xz + yw) \\ 2(xy + zw) & 1 - 2(x^2 + z^2) & 2(yz - xw) \\ 2(xz - yw) & 2(yz + xw) & 1 - 2(x^2 + y^2) \end{bmatrix}$$

Unit quaternion to rotation matrix

Unit Quaternions as 3D Rotations

- Composing rotations using unit quaternions

$$\mathbf{q}_2 = \mathbf{q}_0 \mathbf{q}_1$$

$$\mathbf{R}(\mathbf{q}_2) = \mathbf{R}(\mathbf{q}_0) \mathbf{R}(\mathbf{q}_1)$$

Two Equivalent Quaternions for 3D Rotation

- Multiply -1 to a quaternion

$$q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \frac{\vec{u}}{\|\vec{u}\|}$$

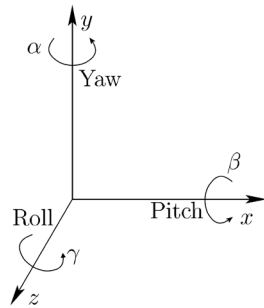
$$-q = \cos \left(\frac{\theta}{2} + \pi \right) + \sin \left(\frac{\theta}{2} + \pi \right) \frac{\vec{u}}{\|\vec{u}\|}$$

- q rotates θ , $-q$ rotates $\theta + 2\pi$

3D Rotation Representations

- Rotation matrix R

- Euler angles



- Axis-angle

- Minimal representation $\omega = \theta \hat{n}$

- Unit quaternion

- Continuous rotations

$$q = w + xi + yj + zk$$

Further Reading

- Section 2.1, Computer Vision, Richard Szeliski
- Quaternion and Rotations, Yan-Bin Jia, <https://graphics.stanford.edu/courses/cs348a-17-winter/Papers/quaternion.pdf>
- Introduction to Robotics, Prof. Wei Zhang, OSU, Lecture 3, Rotational Motion, <http://www2.ece.ohio-state.edu/~zhang/RoboticsClass/index.html>
- [On the Continuity of Rotation Representations in Neural Networks](#). Zhou et al., CVPR, 2019.