Velocity Kinematics: Exponential Coordinates of Rigid-Body Motions and Twists

CS 6341 Robotics

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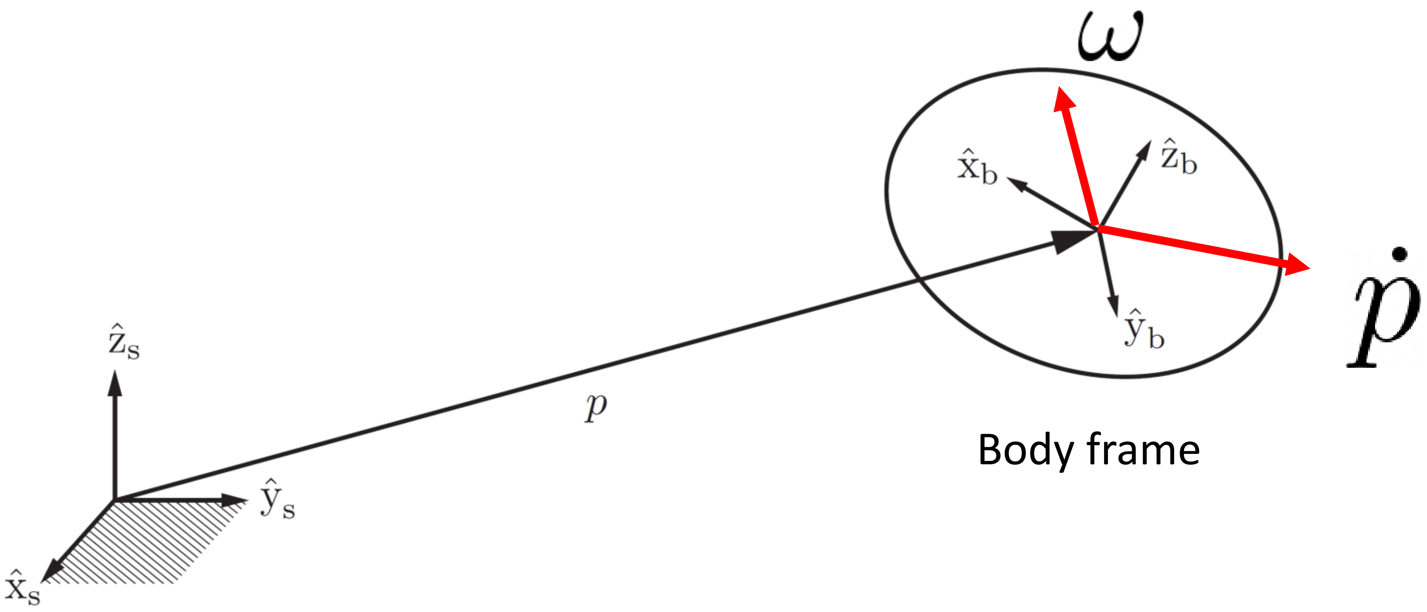
Recall Angular Velocity and Linear Velocity



$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$

$$\dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

- Angular velocity

$$\omega = \hat{\omega}\dot{\theta}$$

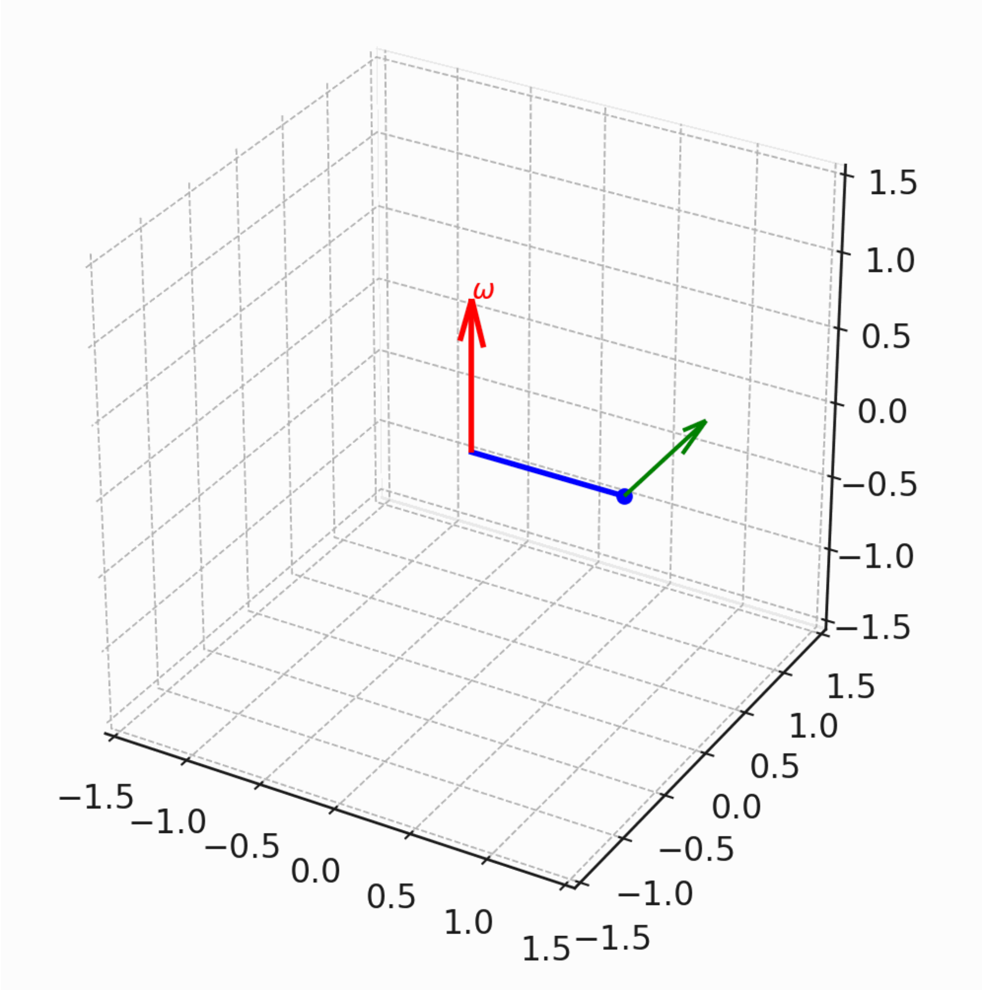
$$\dot{R}R^{-1} = [\omega_s]$$

$$R^{-1}\dot{R} = [\omega_b]$$

- Linear velocity $\dot{p}$

The linear velocity of the origin of {b} expressed in the fixed frame {s}

Angular Velocities



Generated by ChatGPT

- **Red arrow** → angular velocity vector $\boldsymbol{\omega}$
- **Blue line** → rotating body-fixed axis.
- **Green arrow** → instantaneous linear velocity of the blue endpoint ($\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$).

Tangential velocity due to rotation

How to use angular velocity and linear velocity?

$$\dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

- How about this?

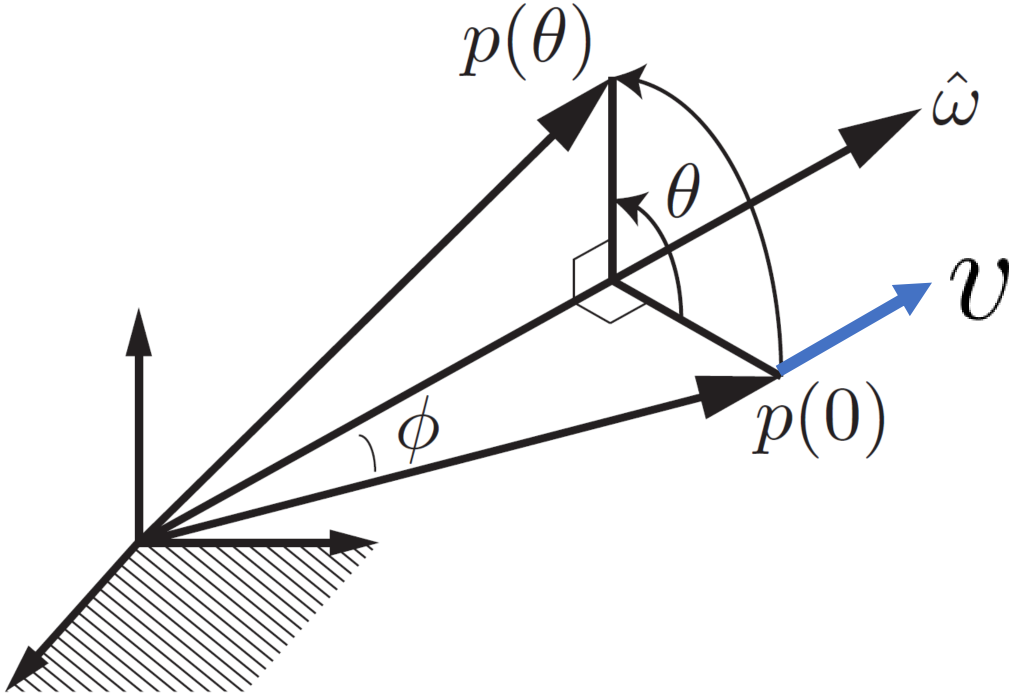
$$T(t + \Delta t) \approx T(t) + \dot{T}(t)\Delta t$$

- Adding $\dot{R}$ directly like this **breaks orthogonality** — the result won't generally be a valid rotation matrix.

- How to compute the transformation after certain time?

Exponential Coordinates of Rigid-Body Motions

- $p(0)$ is rotated to $p(\theta)$
  - At a constant rate of 1 rad/s
- $p(t)$: path traced by the tip of vector

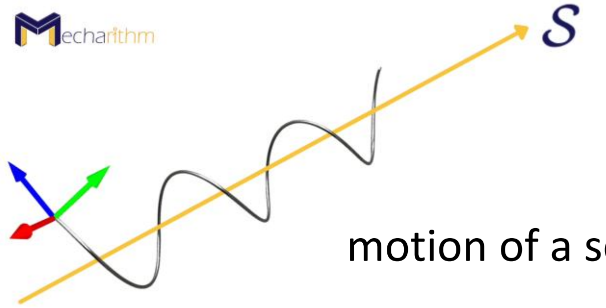


Velocity of the tip vector

$$\dot{p}(t) = v + \hat{\omega} \times p(t)$$

An additional linear velocity

Tangential velocity due to rotation



motion of a screw

Exponential Coordinates of Rigid-Body Motions

- Linear Differential Equations $$\dot{p}(t) = v + \hat{\omega} \times p(t)$$
- Can we solve this equation to compute $p(t)$?
  - After time t, where is the vector?
- A scalar linear differential equation $\dot{x}(t) = ax(t)$ $x(t) \in \mathbb{R},\ a \in \mathbb{R}$

Initial condition $x(0) = x_0$ Solution $x(t) = e^{at}x_0$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

Exponential Coordinates of Rigid-Body Motions

- Vector linear differential equation

$$\dot{x}(t) = Ax(t) \qquad x(t) \in \mathbb{R}^n,\ A \in \mathbb{R}^{n \times n}$$

Initial condition $x(0) = x_0$ Solution $x(t) = e^{At}x_0$

matrix exponential $$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

If A is constant and finite, this series converges to a finite limit

Exponential Coordinates of Rigid-Body Motions

- Linear Differential Equations $\dot{p}(t) = v + \hat{\omega} \times p(t)$
- Can we solve this equation to compute $p(t)$?
- Let's convert it to this form $\dot{x}(t) = Ax(t)$

$$\dot{\tilde{p}}(t) = \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix} \tilde{p}(t) \qquad \tilde{p}(t) = \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \qquad [\mathcal{S}] = \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

Homogenous coordinate

Exponential Coordinates of Rigid-Body Motions

- Linear Differential Equations $\dot{p}(t) = v + \hat{\omega} \times p(t)$

$$\dot{\tilde{p}}(t) = [\mathcal{S}]\tilde{p}(t) \qquad [\mathcal{S}] = \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix}$$

- Solution $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$

Homogenous transformation $T(t) = e^{[\mathcal{S}]t}$

What is this??

matrix exponential $e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$

How to compute a 4 x 4 matrix from this?

Exponential Coordinates of Rigid-Body Motions

$$[\mathcal{S}] = \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix}$$

- Let's do some computation $T(t) = e^{[\mathcal{S}]t}$
- We can change t to θ

$$T(\theta) = e^{[\mathcal{S}]\theta} = I + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \cdots$$

$$= I + \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix}\theta + \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix}^2\frac{\theta^2}{2!} + \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix}^3\frac{\theta^3}{3!} + \cdots$$

$$= I + \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix}\theta + \begin{bmatrix} [\hat{\omega}]^2 & [\hat{\omega}]v \\ 0 & 0 \end{bmatrix}\frac{\theta^2}{2!} + \begin{bmatrix} [\hat{\omega}]^3 & [\hat{\omega}]^2 v \\ 0 & 0 \end{bmatrix}\frac{\theta^3}{3!} + \cdots$$

Exponential Coordinates of Rigid-Body Motions

$$T(\theta) = e^{[\mathcal{S}]\theta} = \begin{bmatrix} R(\theta) & G(\theta)v \\ 0 & 1 \end{bmatrix}$$

$$R(\theta) = I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots$$

$$G(\theta) = I\theta + [\hat{\omega}] \frac{\theta^2}{2!} + [\hat{\omega}]^2 \frac{\theta^3}{3!} + \cdots$$

We have $[\hat{\omega}]^3 = -[\hat{\omega}]$

Exponential Coordinates of Rigid-Body Motions

$$R(\theta) = I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots$$

$$= \quad I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat{\omega}]^2$$

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \end{aligned}$$

$$R(\theta) = e^{[\hat{\omega}]\theta}$$

$$R(\theta) = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$

Rodrigues' formula: exponential coordinates to rotation matrix

Exponential Coordinates of Rigid-Body Motions

$$[\hat{\omega}]^3 = -[\hat{\omega}]$$

$$G(\theta) = I\theta + [\hat{\omega}]\frac{\theta^2}{2!} + [\hat{\omega}]^2\frac{\theta^3}{3!} + \cdots$$

$$= I\theta + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat{\omega}] + \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \cdots\right)[\hat{\omega}]^2$$

$$= I\theta + (1 - \cos\theta)[\hat{\omega}] + (\theta - \sin\theta)[\hat{\omega}]^2$$

Exponential Coordinates of Rigid-Body Motions

$$T(\theta) = e^{[\mathcal{S}]\theta} = \begin{bmatrix} R(\theta) & G(\theta)v \\ 0 & 1 \end{bmatrix} \qquad [\mathcal{S}] = \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 & \left(I\theta + (1-\cos\theta)[\hat{\omega}] + (\theta - \sin\theta)[\hat{\omega}]^2\right)v \\ 0 & 1 \end{bmatrix}$$

Conclusion: given unit angular velocity $\hat{\omega}$ and linear velocity v

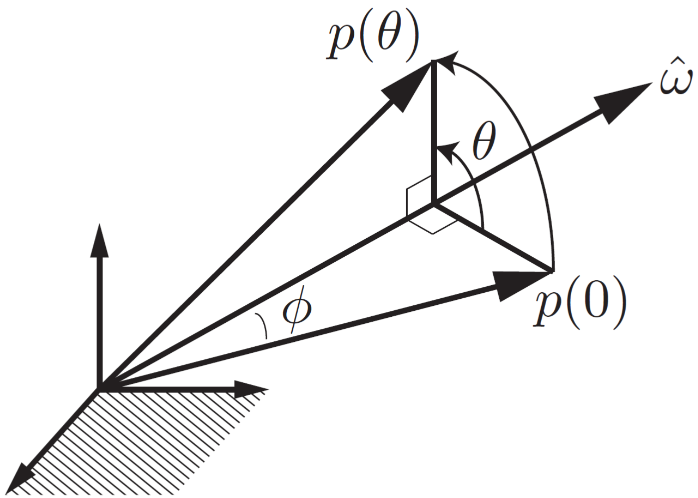
Use the above equation to compute the homogenous transformation after θ

If $\|\omega\| = 1$

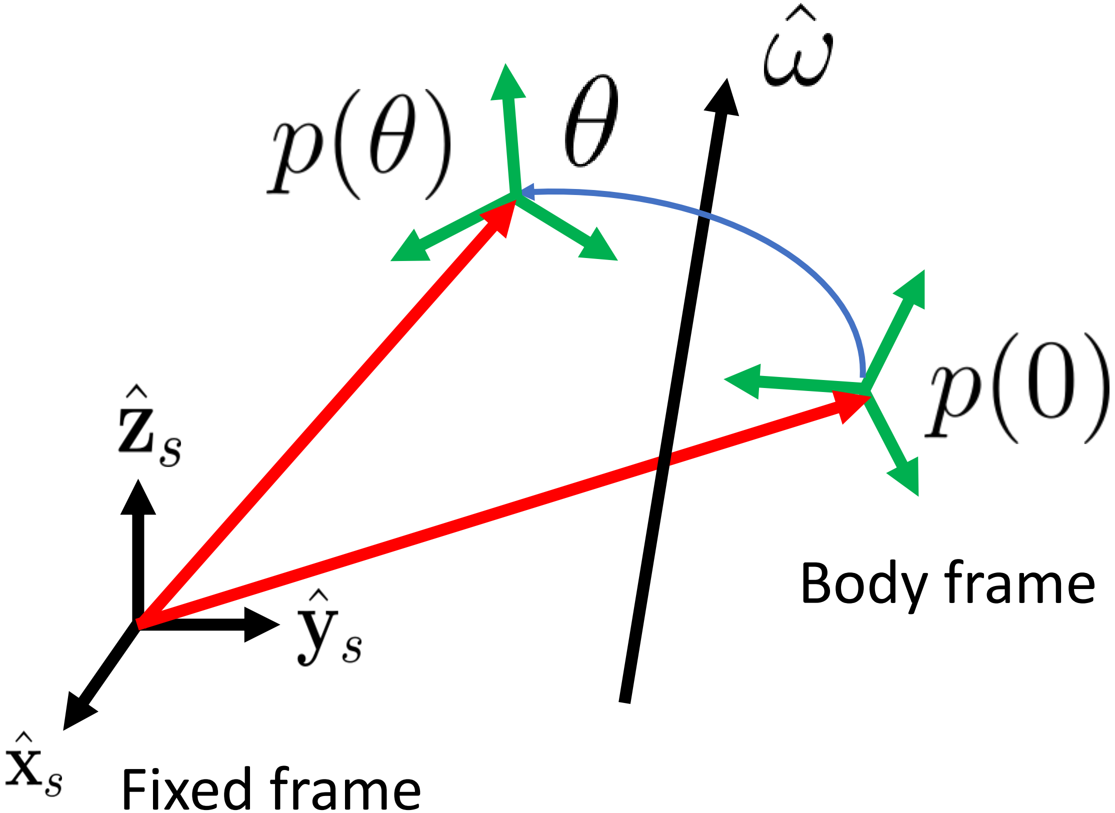
If $\omega = 0$ and $\|v\| = 1$

$$T(\theta) = e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$$

What if the rotation axis not going through the origin?



Case 1 (so far)



Case 2

What if the rotation axis not going through the origin?

- $p(0)$ is rotated to $p(\theta)$
  - At a constant rate of 1 rad/s

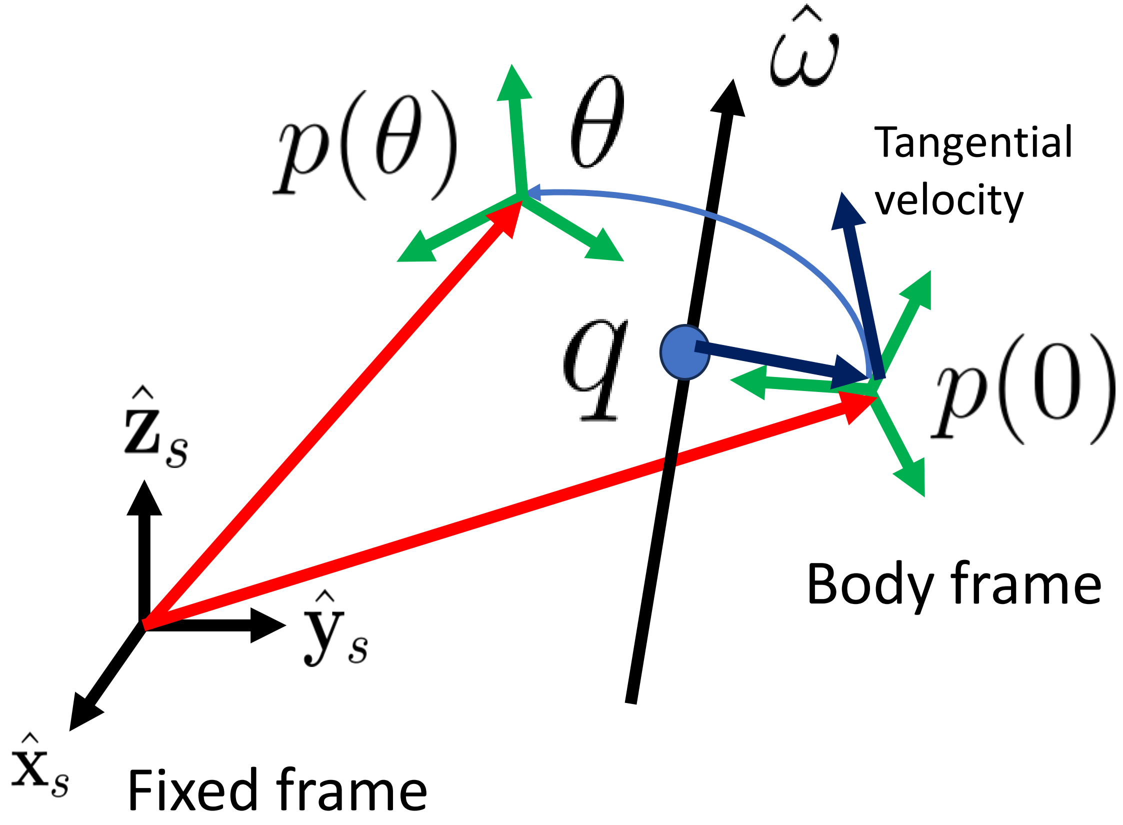
- $p(t)$: path traced by the tip of vector

  Velocity of the tip vector

$$\dot{p}(t) = v + \hat{\omega} \times (p(t) - q)$$

An additional linear velocity

Tangential velocity due to rotation



$$\dot{p}(t) = -\hat{\omega} \times q + v + \hat{\omega} \times p(t)$$

What if the rotation axis not going through the origin?

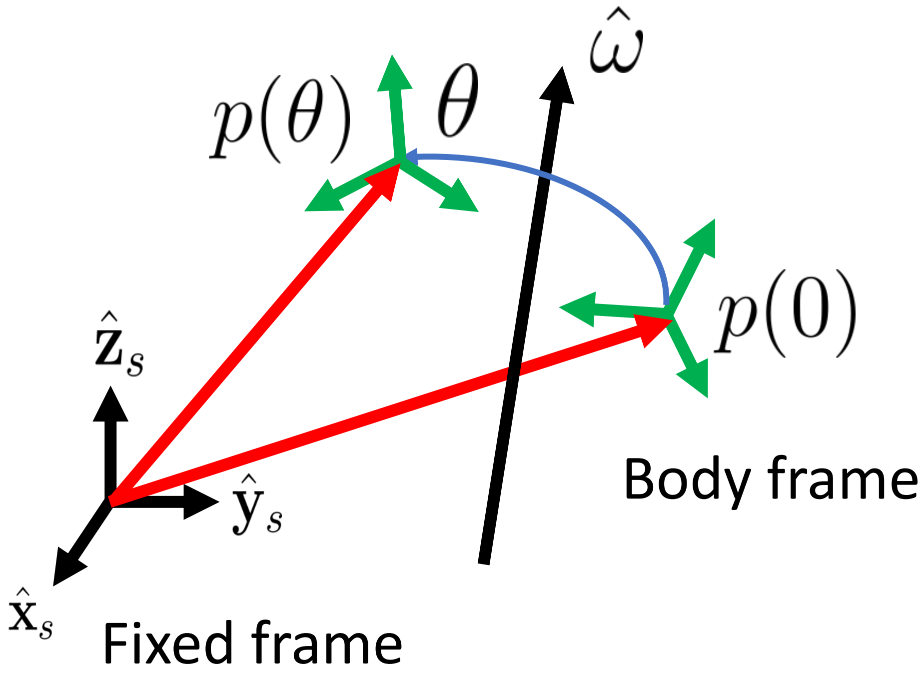
$$\dot{p}(t) = -\hat{\omega} \times q + v + \hat{\omega} \times p(t)$$

- Angular velocity $\hat{\omega}$
- Linear velocity

$$v_1 = -\hat{\omega} \times q + v$$

q can be any point on the rotation axis

$$T(\theta) = e^{[\mathcal{S}]\theta} \qquad [\mathcal{S}] = \begin{bmatrix} [\hat{\omega}] & v_1 \\ 0 & 0 \end{bmatrix}$$



Twist

- Let’s combine angular velocity and linear velocity into a 6D vector called twist

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

- Twist can be defined in fixed frame or body frame

Spatial twist $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6$

Body twist $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$

Relationship between Spatial Twist and Body Twist

- For angular velocity $$\omega_s = R\omega_b$$ $$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

- For linear velocity

$$v_s = -\omega_s \times q_s + v_s^0$$

$$v_b = -\omega_b \times q_b + v_b^0$$

An additional linear velocity

$$q_s = Rq_b + p \qquad v_s^0 = Rv_b^0$$

Relationship between Spatial Twist and Body Twist

- Some derivation

$$
\begin{aligned}
v_s &= -\omega_s \times q_s + v_s^0 \\
&= -[R\omega_b](Rq_b + p) + Rv_b^0 \\
&= -R[\omega_b]R^T(Rq_b + p) + Rv_b^0 \qquad R[\omega]R^{\mathrm{T}} = [R\omega] \\
&= R(-[\omega_b]q_b + v_b^0) - R[\omega_b]R^T p \\
&= Rv_b - [R\omega_b]p \\
&= [p]R\omega_b + Rv_b \qquad [\omega]p = -[p]\omega
\end{aligned}
$$

Relationship between Spatial Twist and Body Twist

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$\downarrow$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

The adjoint representation of $T = (R, p) \in SE(3)$ $\quad \mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$

Relationship between Spatial Twist and Body Twist

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$$

$$R[\omega]R^{\mathrm{T}} = [R\omega]$$

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s$$

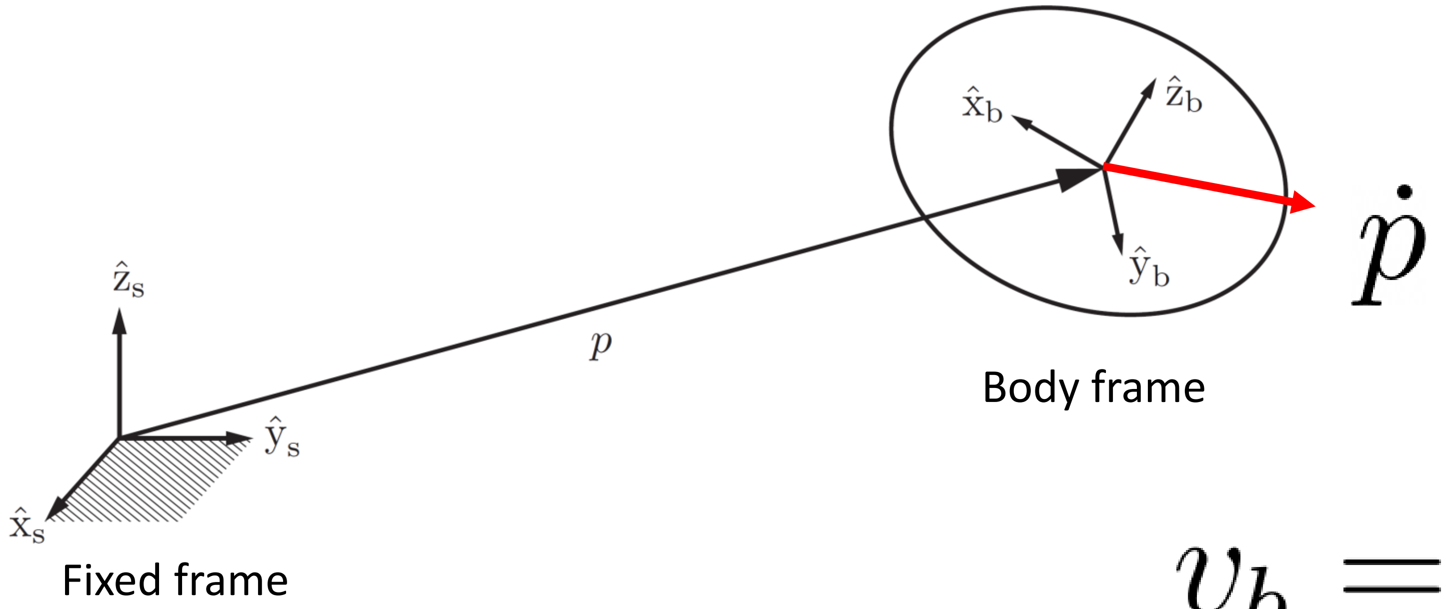
In general

$$\mathcal{V}_c = [\mathrm{Ad}_{T_{cd}}]\mathcal{V}_d, \qquad \mathcal{V}_d = [\mathrm{Ad}_{T_{dc}}]\mathcal{V}_c$$

Relationship between Spatial Twist and Body Twist

- What is the relationship between v_s, v_b and $\dot{p}$?

$$\dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$



- Linear velocity $\dot{p}$

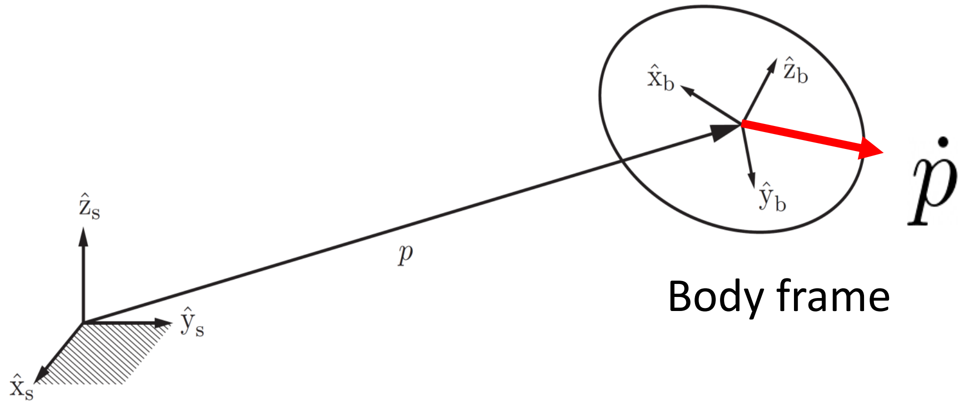
The linear velocity of the origin of {b} expressed in the fixed frame {s}

$$v_b = R_{bs}\dot{p} = R^T\dot{p}$$

Relationship between Spatial Twist and Body Twist

- What is the relationship between v_s, v_b and $\dot{p}$?

$$\dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$



- Linear velocity $\dot{p}$

The linear velocity of the origin of {b} expressed in the fixed frame {s}

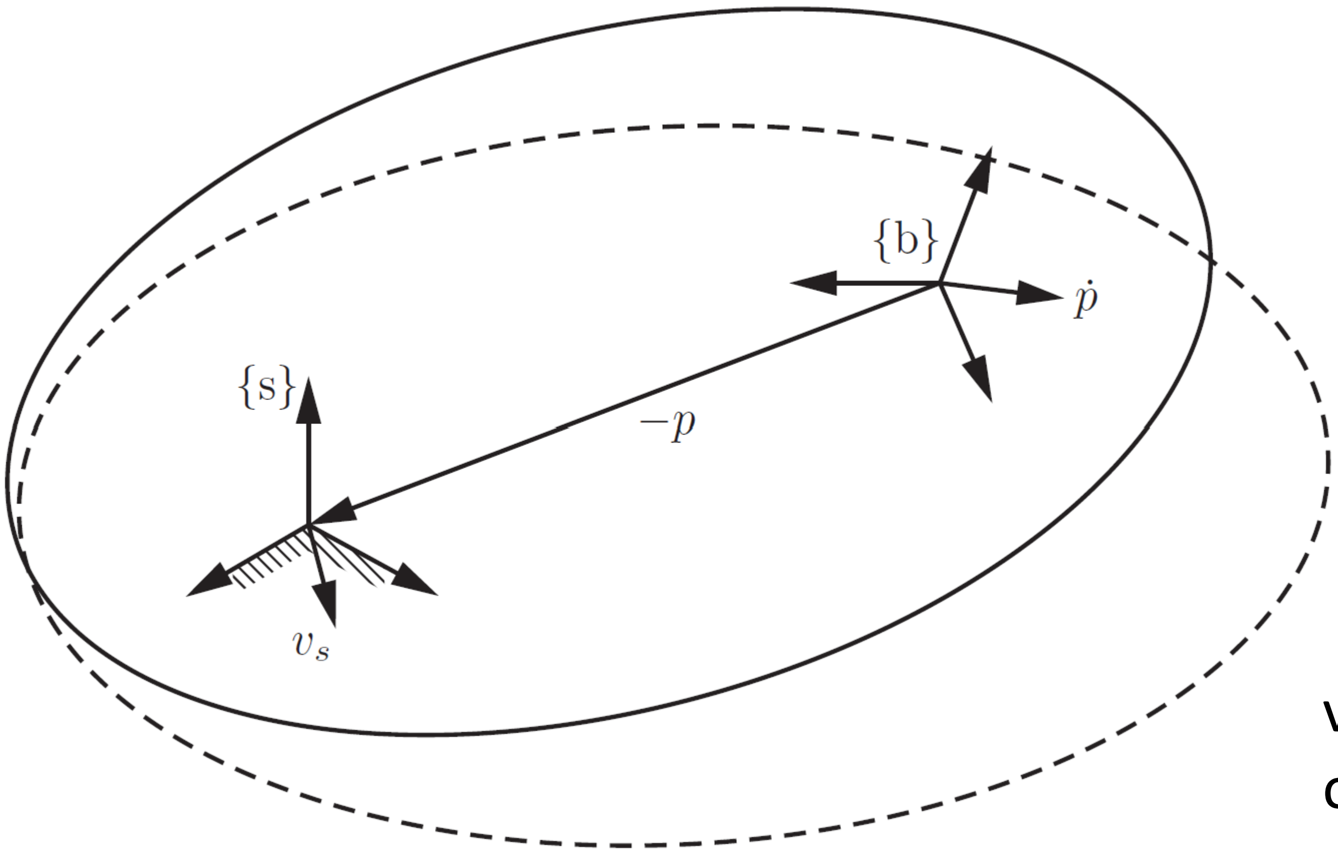
$$v_b = R_{bs}\dot{p} = R^T\dot{p}$$

$$v_s = [p]R\omega_b + Rv_b = [p]\omega_s + RR^T\dot{p}$$

$$v_s = \dot{p} + \omega_s \times (-p)$$

$$[\omega]p = -[p]\omega$$

Relationship between Spatial Twist and Body Twist



$$v_s = \dot{p} + \omega_s \times (-p)$$

v_s is the instantaneous velocity of the point on this body currently at the fixed-frame origin, expressed in the fixed frame

Summary

- Exponential Coordinates of Rigid-Body Motions

- Twists
  - Spatial twists

  - Body twists

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017