Velocity Kinematics: Angular Velocity and Linear Velocity

CS 6341 Robotics

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Forward Kinematics

- Forward kinematics of a robot: calculation of the position and orientation of its end-effector from its joint coordinates

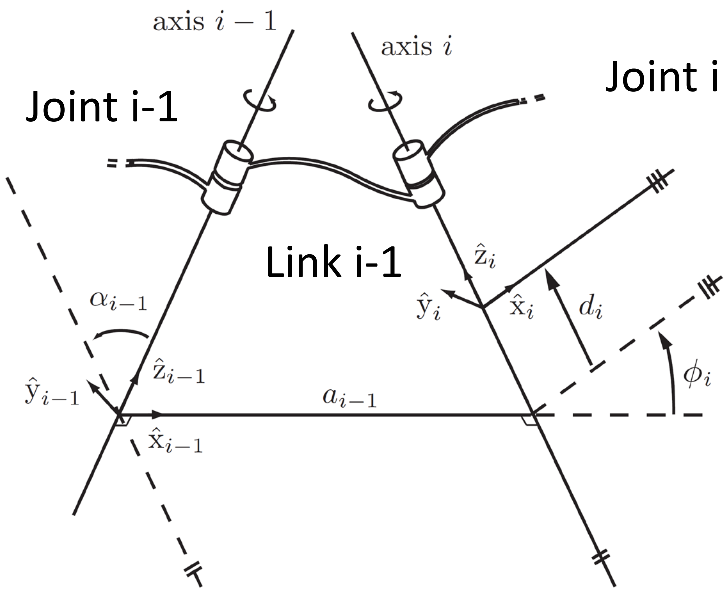
End-effector transformation

Joint coordinates θ $\Rightarrow$ $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

Forward Kinematics with D-H Parameters

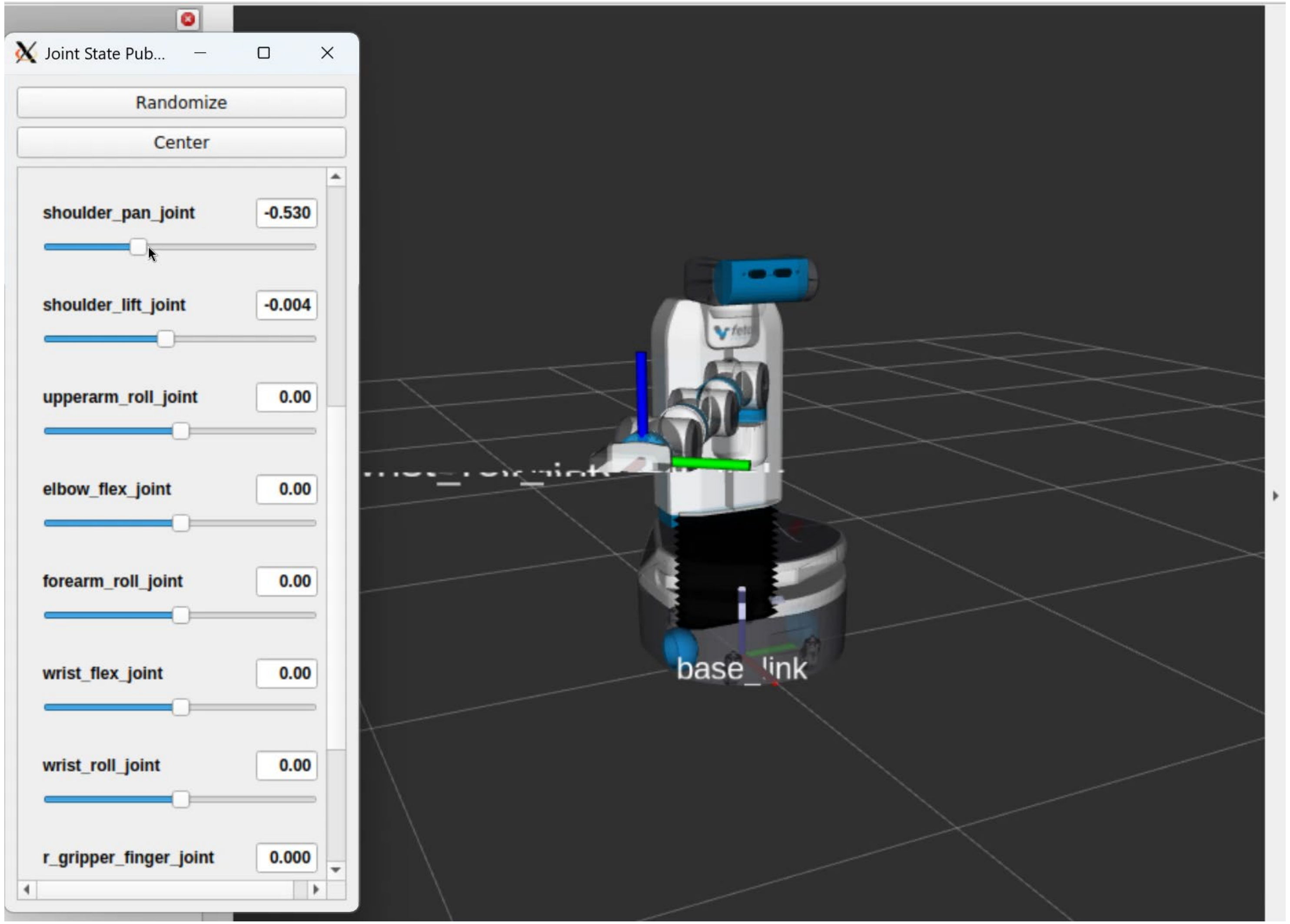
- Link frame transformation

$$T_{i-1,i} \quad = \quad \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Trans}(\hat{\mathrm{z}}, d_i)\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i)$$



$$T_{0n}(\theta_1, \ldots, \theta_n) = T_{01}(\theta_1)T_{12}(\theta_2)\cdots T_{n-1,n}(\theta_n)$$

What is the Velocity of the End-effector?



Why we need to care about end-effector velocity?

https://www.youtube.com/watch?v=wXxrmussq4E

Velocity Kinematics

- Given joint positions and velocities $\theta \in \mathbb{R}^n$ $\dot{\theta}$

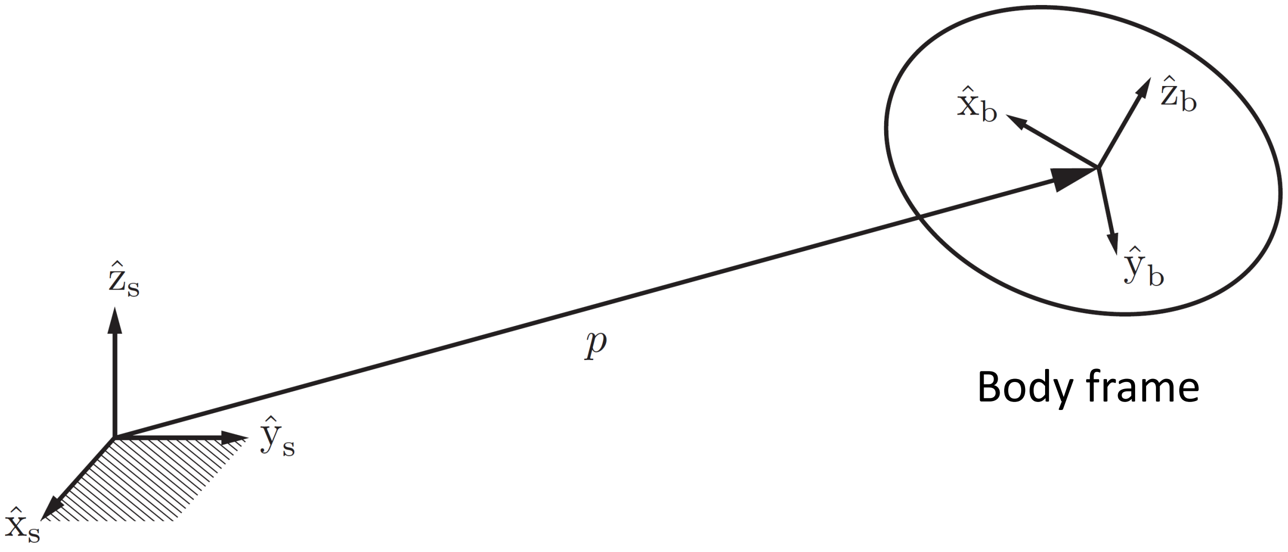
- Compute the velocity of the end-effector

End-effector configuration

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \xrightarrow[\dot{p}(t) = \frac{d}{dt}p(t)]{\dot{R}(t) = \frac{d}{dt}R(t)} \dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

What is this?

Recall Rigid-Body in 3D



- Origin of the body frame

$$p = p_1\hat{x}_s + p_2\hat{y}_s + p_3\hat{z}_s$$

- Axes of the body frame

$$\begin{aligned}\hat{x}_b &= r_{11}\hat{x}_s + r_{21}\hat{y}_s + r_{31}\hat{z}_s, \\ \hat{y}_b &= r_{12}\hat{x}_s + r_{22}\hat{y}_s + r_{32}\hat{z}_s, \\ \hat{z}_b &= r_{13}\hat{x}_s + r_{23}\hat{y}_s + r_{33}\hat{z}_s.\end{aligned}$$

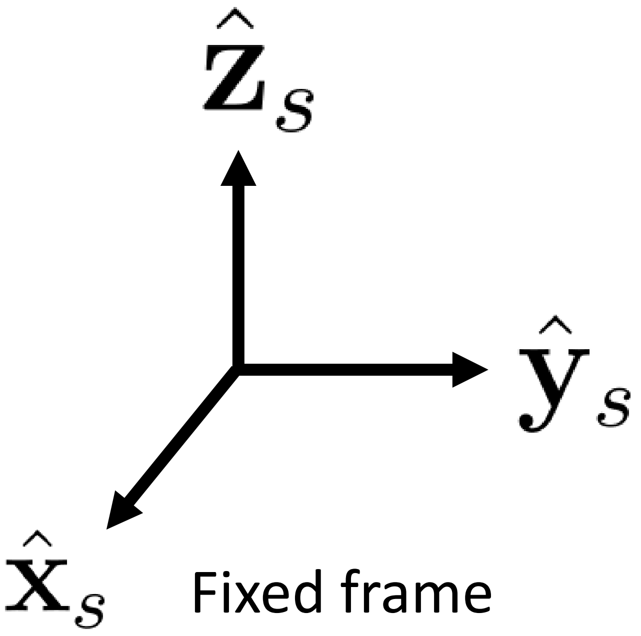
Translation $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

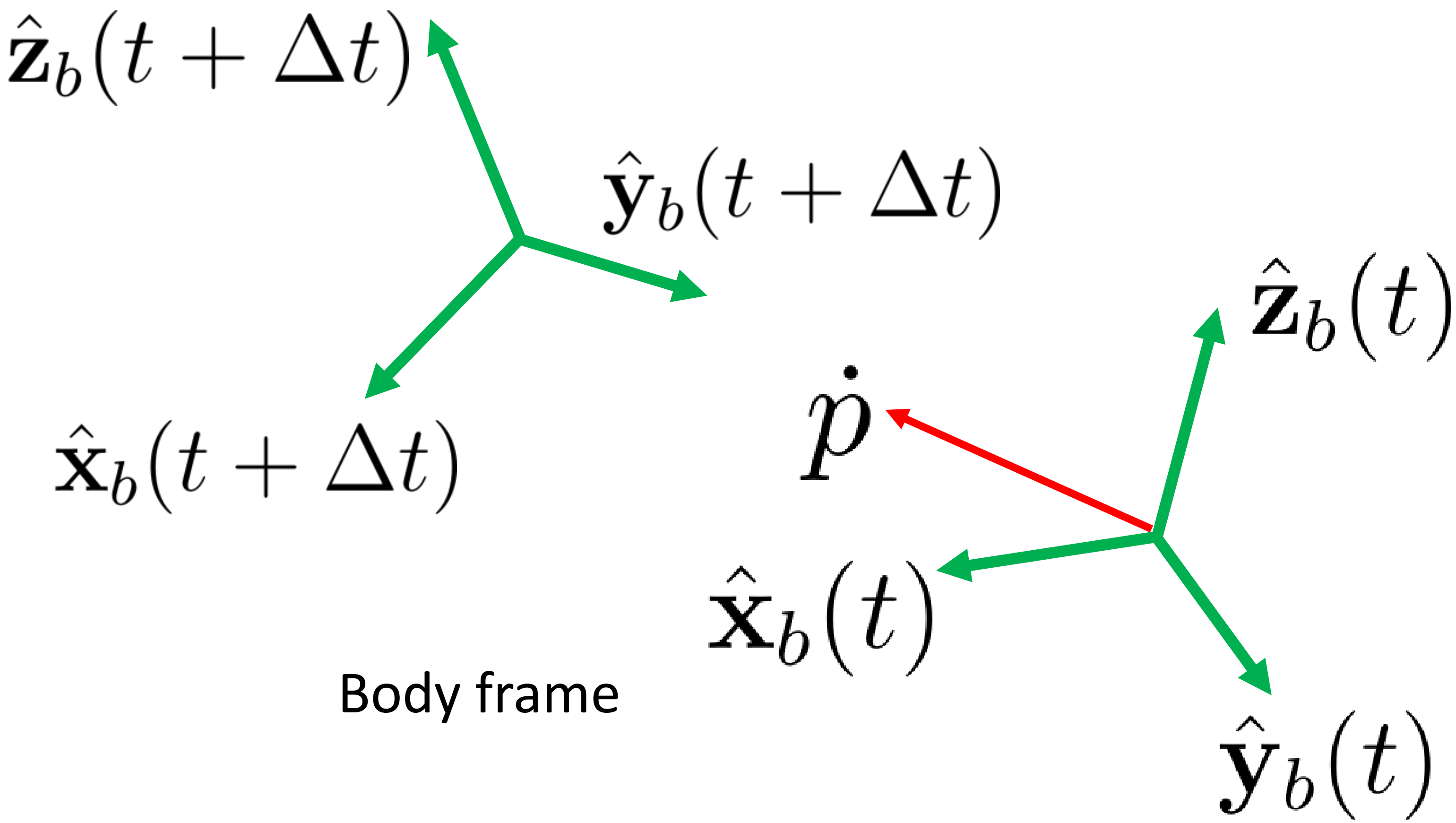
Rotation matrix

$$R = [\hat{x}_b \;\; \hat{y}_b \;\; \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Meanings of the column vectors

Angular Velocity and Linear Velocity





- Linear velocity $\dot{p}$

  The linear velocity of the origin of {b} expressed in the fixed frame {s}

- How about $\dot{R}$?

Recall Rotating a Vector or a Frame

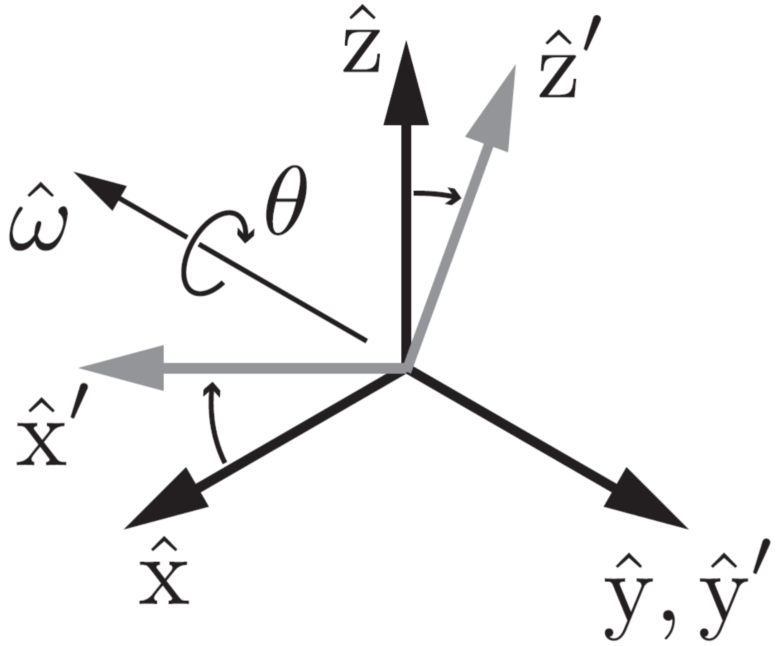
- Rotate frame {c} about a unit axis $\hat{\omega}$ by θ to get frame {c'}, {c} is aligned with {s} in the beginning

$$R = R_{sc'}$$

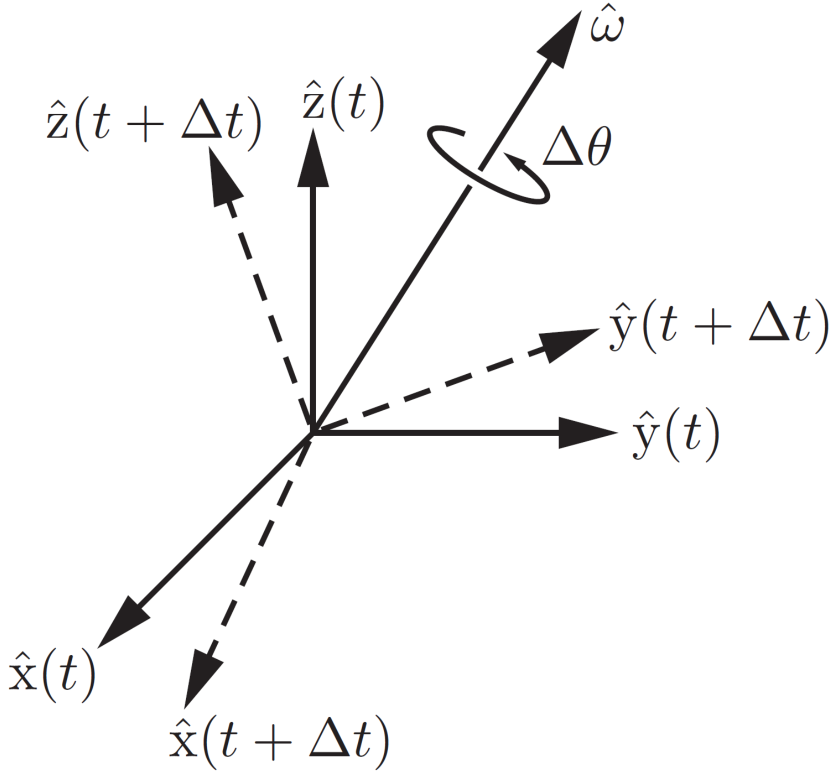
frame {c'} relative to frame {s}

- Rotation operation

$$R = \mathrm{Rot}(\hat{\omega}, \theta)$$



Angular Velocities



- Axes $\{\hat{x}, \hat{y}, \hat{z}\}$ Unit length

  Rotating around $\hat{\omega}$ by $\Delta\theta$

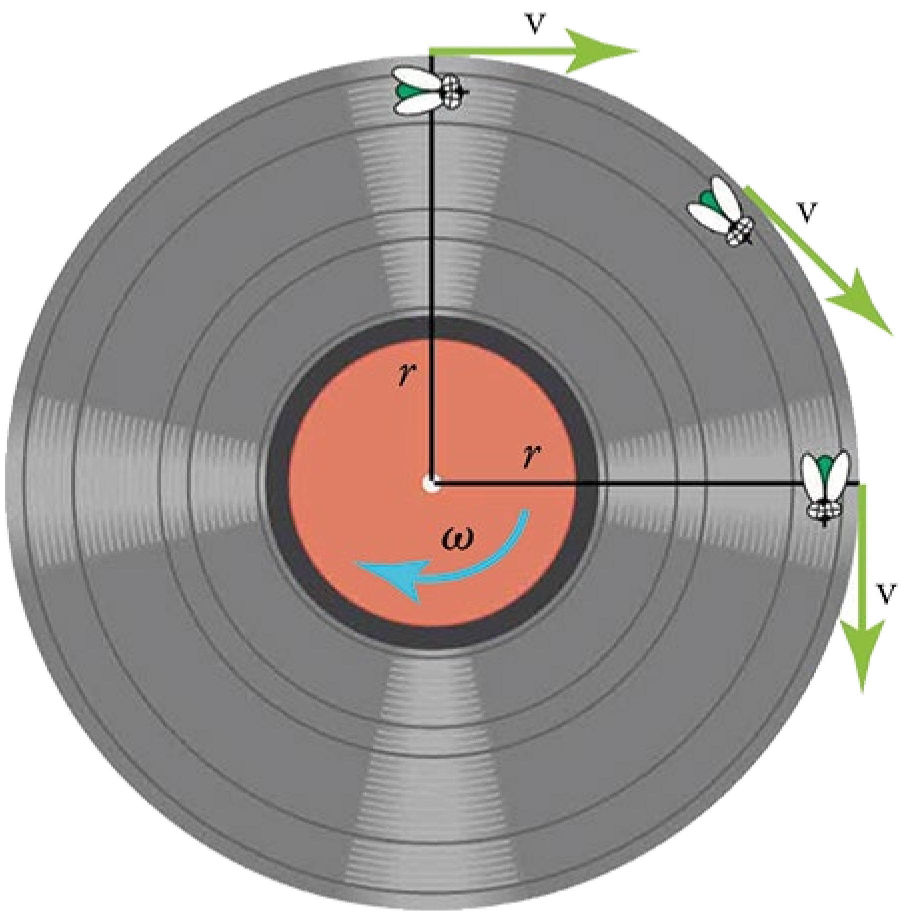
  $\hat{\omega}$ is coordinate free for now

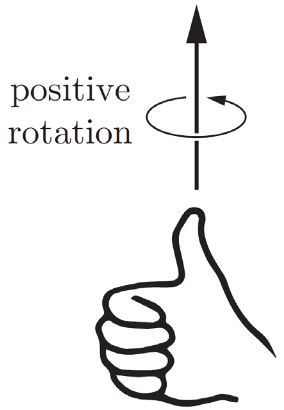
  $\Delta t \to 0 \quad \Delta\theta/\Delta t \longrightarrow \dot{\theta}$ **Instantaneous velocity**

  $\hat{\omega}$ instantaneous axis of rotation

- **Definition** Angular velocity $\omega = \hat{\omega}\dot{\theta}$

Angular Velocity and Tangential Velocity

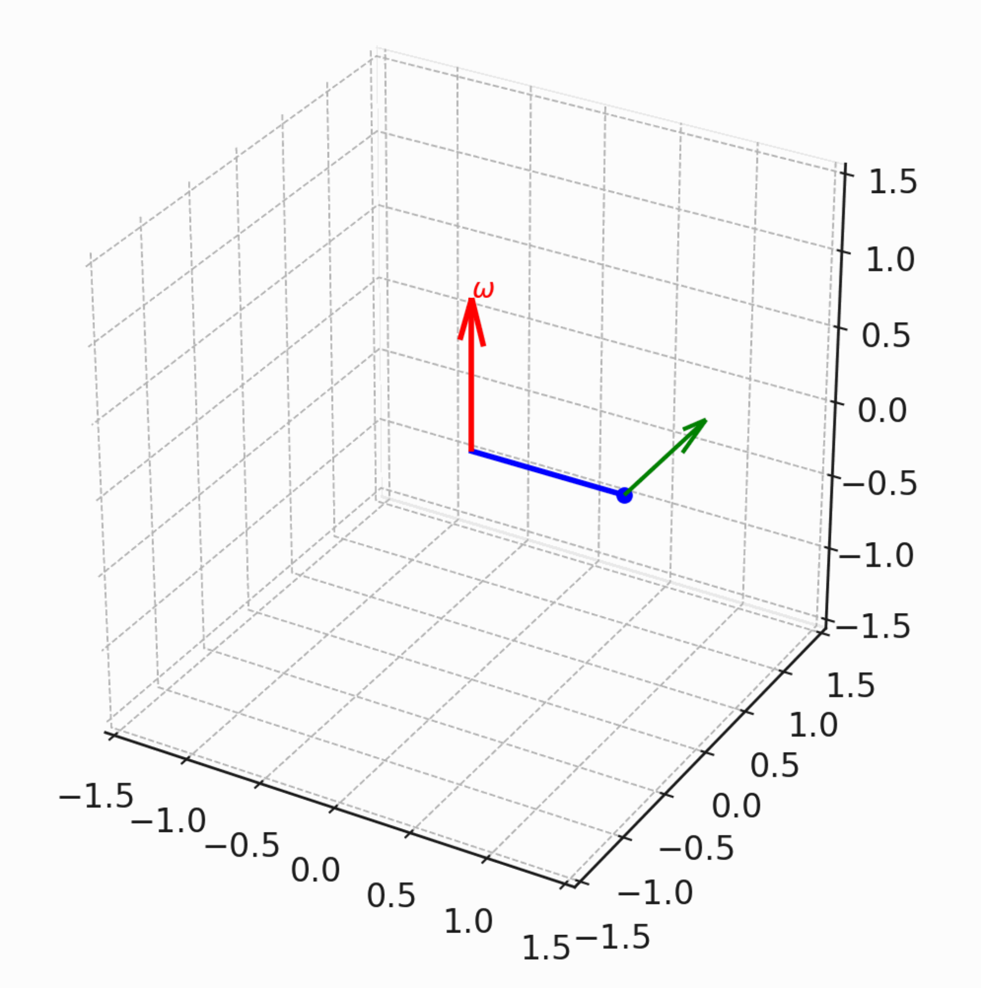




Speed v = r w

https://openstax.org/books/physics/pages/6-1-angle-of-rotation-and-angular-velocity

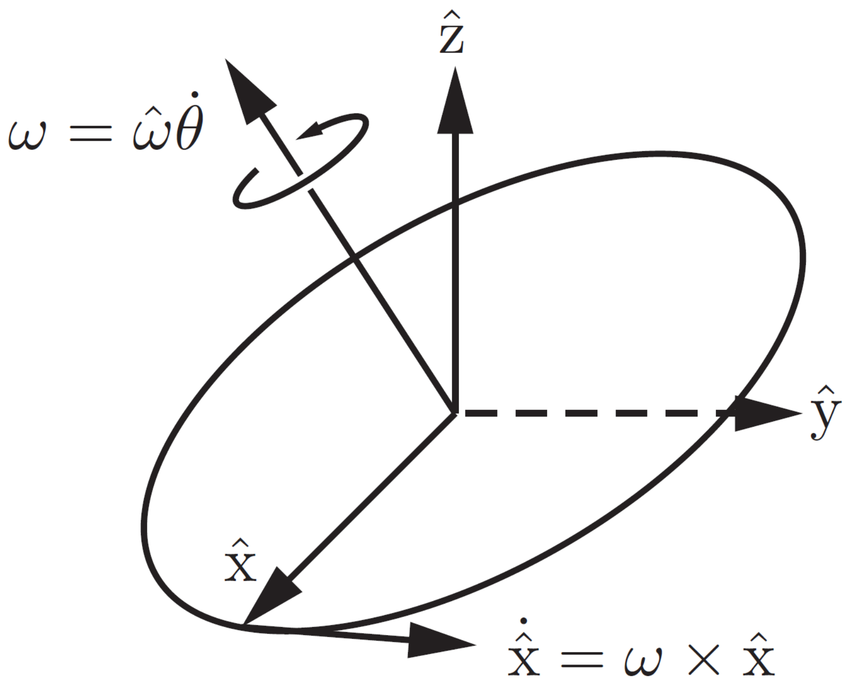
Angular Velocities



Generated by ChatGPT

- **Red arrow** → angular velocity vector $\boldsymbol{\omega}$
- **Blue line** → rotating body-fixed axis.
- **Green arrow** → instantaneous linear velocity of the blue endpoint ($\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$).

Angular Velocities



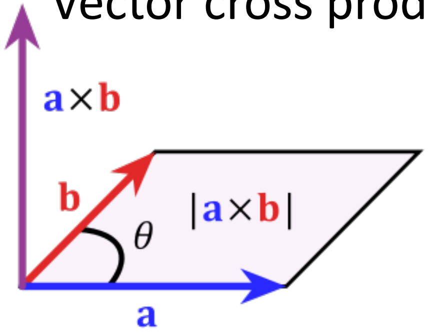
- Angular velocity $\omega = \hat{\omega}\dot{\theta}$
- Compute time derivates of these axes caused by rotation $\dot{\hat{\mathrm{x}}}$ (tangential velocity)

$$\dot{\hat{\mathbf{x}}} = \omega \times \hat{\mathbf{x}}$$

$$\dot{\hat{\mathbf{y}}} = \omega \times \hat{\mathbf{y}}$$

$$\dot{\hat{\mathbf{z}}} = \omega \times \hat{\mathbf{z}}$$

Vector cross product

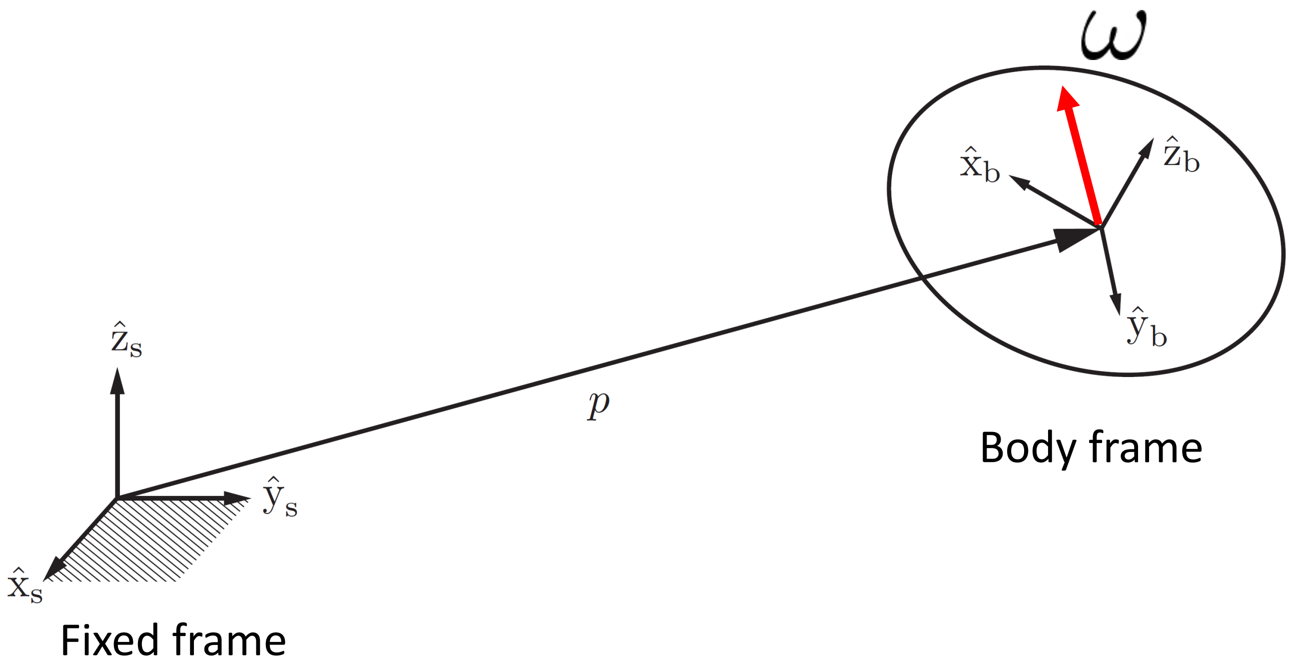


$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \, \mathbf{n}$$

https://en.wikipedia.org/wiki/Cross_product

Angular Velocities

- To express these equations in coordinates, we must choose a reference frame for ω
  - Two natural choices: fixed frame {s} or body frame {b}



Angular Velocities in Fixed Frame

- Consider fixed frame {s}
  - Orientation of the body frame at time t $R(t) = [\hat{\mathrm{x}}_{\mathrm{b}} \ \ \hat{\mathrm{y}}_{\mathrm{b}} \ \ \hat{\mathrm{z}}_{\mathrm{b}}]$
    $= [r_1(t) \ r_2(t) \ r_3(t)]$
  - Time rate of change $\dot{R}(t)$
  - Angular velocity $\omega_s \in \mathbb{R}^3$

$$
\begin{aligned}
\dot{\hat{\mathrm{x}}} &= \mathrm{w} \times \hat{\mathrm{x}}, \\
\dot{\hat{\mathrm{y}}} &= \mathrm{w} \times \hat{\mathrm{y}}, \\
\dot{\hat{\mathrm{z}}} &= \mathrm{w} \times \hat{\mathrm{z}}.
\end{aligned}
$$

$$\dot{r}_i = \omega_s \times r_i, \qquad i = 1, 2, 3.$$

Column

$$\dot{R} = [\omega_s \times r_1 \ \ \omega_s \times r_2 \ \ \omega_s \times r_3] = \omega_s \times R.$$

Skew-symmetric Matrix

https://en.wikipedia.org/wiki/Skew-symmetric_matrix

$$x = [x_1 \ x_2 \ x_3]^{\mathrm{T}} \in \mathbb{R}^3$$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$$[x] = -[x]^{\mathrm{T}}$$

$$\omega_s \times R = [\omega_s]R$$

$$\dot{R} = \omega_s \times R$$

$$[\omega_s]R = \dot{R}$$

$$[\omega_s] = \dot{R}R^{-1}$$

Skew-symmetric Matrix

Proposition $$R[\omega]R^{\mathrm{T}} = [R\omega] \quad \omega \in \mathbb{R}^3 \quad R \in SO(3)$$

See Lynch & Park for proof

Proof. Letting r_i^{T} be the ith row of R, we have

$$\begin{aligned}
R[\omega]R^{\mathrm{T}} &= \begin{bmatrix} r_1^{\mathrm{T}}(\omega \times r_1) & r_1^{\mathrm{T}}(\omega \times r_2) & r_1^{\mathrm{T}}(\omega \times r_3) \\ r_2^{\mathrm{T}}(\omega \times r_1) & r_2^{\mathrm{T}}(\omega \times r_2) & r_2^{\mathrm{T}}(\omega \times r_3) \\ r_3^{\mathrm{T}}(\omega \times r_1) & r_3^{\mathrm{T}}(\omega \times r_2) & r_3^{\mathrm{T}}(\omega \times r_3) \end{bmatrix} \\
&= \begin{bmatrix} 0 & -r_3^{\mathrm{T}}\omega & r_2^{\mathrm{T}}\omega \\ r_3^{\mathrm{T}}\omega & 0 & -r_1^{\mathrm{T}}\omega \\ -r_2^{\mathrm{T}}\omega & r_1^{\mathrm{T}}\omega & 0 \end{bmatrix} \\
&= [R\omega],
\end{aligned}$$

Angular Velocities in Body Frame

- Consider body frame {b} ω_b

Change of reference frame $\omega_s = R_{sb}\omega_b$

$$\omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^{\mathrm{T}}\omega_s$$

$$\begin{aligned}[\omega_b] &= [R^{\mathrm{T}}\omega_s] \\ &= R^{\mathrm{T}}[\omega_s]R \quad \text{(proposition)} \\ &= R^{\mathrm{T}}(\dot{R}R^{\mathrm{T}})R \\ &= R^{\mathrm{T}}\dot{R} = R^{-1}\dot{R}\end{aligned}$$

$$[\omega_s] = \dot{R}R^{-1}$$

Angular Velocities

- Orientation of the body frame at time t in the fixed frame $R(t)$

$$R_{sb}(t)$$

- Angular velocity w

$$\begin{aligned} \dot{R}R^{-1} &= [\omega_s] \\ R^{-1}\dot{R} &= [\omega_b] \end{aligned}$$

- Change of reference frame of angular velocity

$$\omega_c = R_{cd}\omega_d$$

Velocity Kinematics

- Compute the velocity of the end-effector

End-effector configuration

What is this?

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \longrightarrow \dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

- $\dot{p}$ linear velocity of the origin of {b} expressed in the fixed frame {s}

$$\dot{R} = [\omega_s]R \qquad \dot{R} = R[\omega_b]$$

Related to angular velocity

- Velocity kinematics: how to compute linear velocity and angular velocity given joint positions and velocities? (future lectures)

Summary

- Velocity Kinematics
- Linear Velocity
- Angular velocity

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017