Homogeneous Transformation Matrices

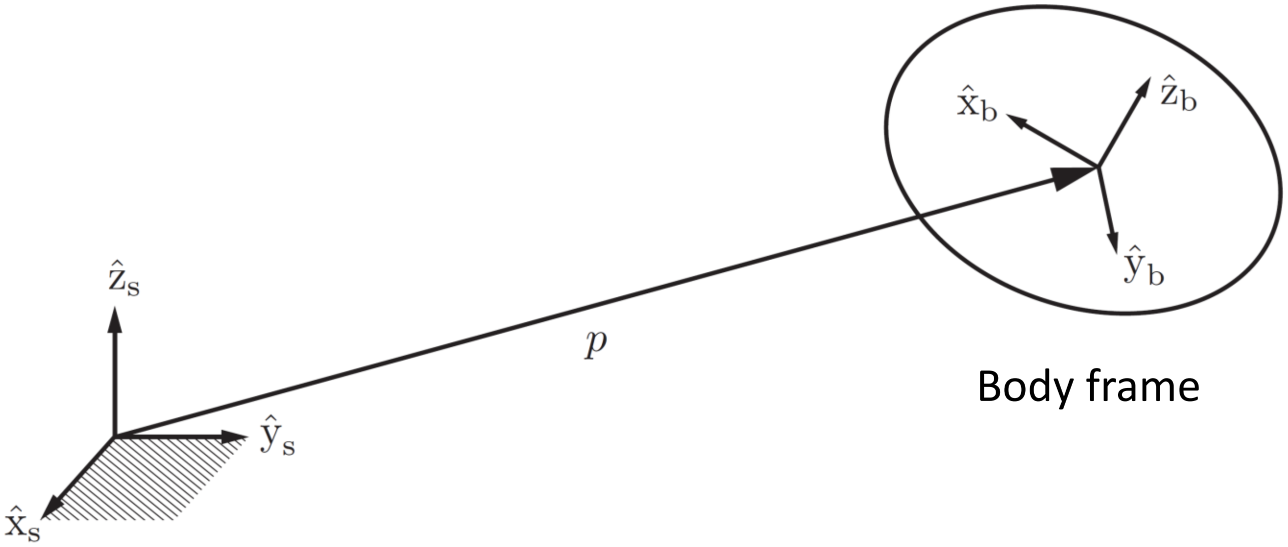
CS 6341 Robotics

Professor Yu Xiang

The University of Texas at Dallas

Rigid-Body in 3D

Translation $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$



Body frame

Fixed frame

$$R = [\hat{\mathrm{x}}_{\mathrm{b}} \;\; \hat{\mathrm{y}}_{\mathrm{b}} \;\; \hat{\mathrm{z}}_{\mathrm{b}}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

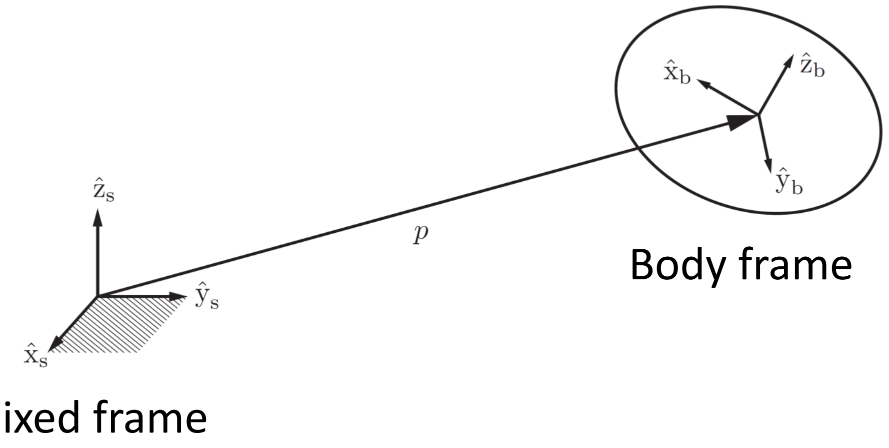
Rotation matrix

An example

$$p = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation Matrices

- Consider body frame {b} in a fixed frame {s}
  - 3D rotation $R \in SO(3)$
  - 3D position $p \in \mathbb{R}^3$



- Special Euclidean group SE(3) or homogenous transformation matrices

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Properties of Transformation Matrices

- The inverse of a transformation matrix is also a transformation matrix

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$$

- Closure $T_1 T_2$
- Associativity $(T_1 T_2) T_3 = T_1 (T_2 T_3)$
- Identity element: identity matrix I
- Not commutative $T_1 T_2 \neq T_2 T_1$

Homogeneous Coordinates

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = w \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene coordinates

Up to scale

Conversion

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Homogeneous Coordinates

$$T\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$$

Homogeneous transformation

Homogeneous coordinates

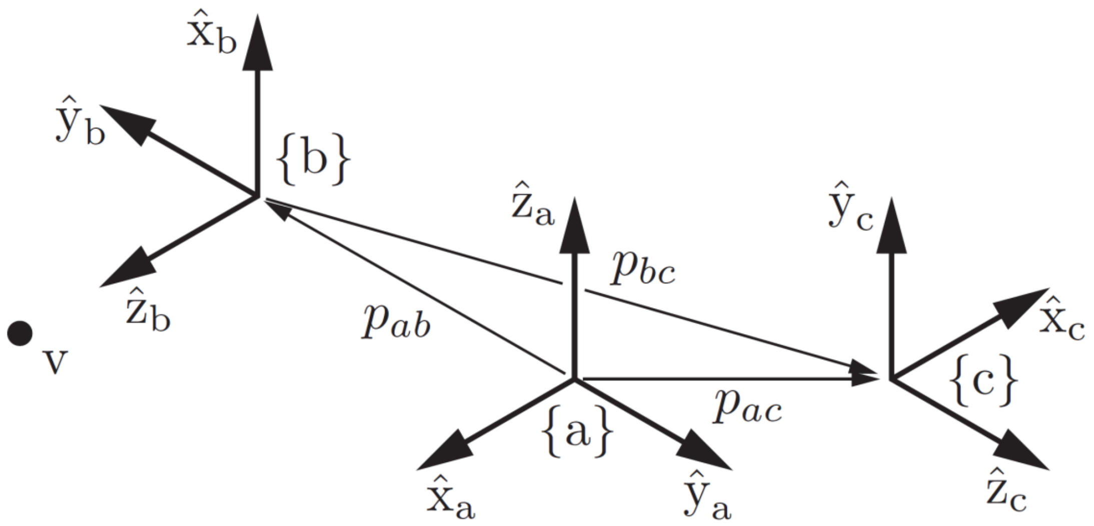
Uses of Transformation Matrices

- Represent the configuration of a rigid-body

- Change the reference frame

- Displace a vector or a frame

Representing a Configuration



$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$T_{sa} = (R_{sa}, p_{sa}) \quad T_{sb} = (R_{sb}, p_{sb})$$

$$T_{sc} = (R_{sc}, p_{sc})$$

$$T_{bc} = (R_{bc}, p_{bc}) \quad R_{bc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$p_{bc} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix} \qquad T_{de} = T_{ed}^{-1}$$

Changing the Reference Frame

$$T_{ab}T_{bc} = T_{a\cancel{b}}T_{\cancel{b}c} = T_{ac}$$

$$T_{ab}v_b = T_{a\cancel{b}}v_{\cancel{b}} = v_a$$

Displacing a Vector or a Frame

- Rotating and then translating $(R, p) = (\mathrm{Rot}(\hat{\omega}, \theta), p)$

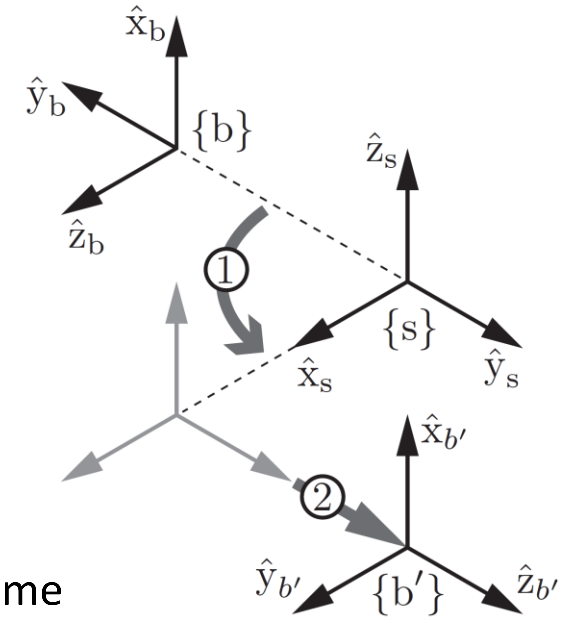
- Transformation matrices

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathrm{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Displacing a Vector or a Frame in Fixed Frame

$$T_{sb'} = TT_{sb} = \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega}, \theta)T_{sb} \qquad \text{(fixed frame)}$$

$$= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & Rp_{sb} + p \\ 0 & 1 \end{bmatrix}$$



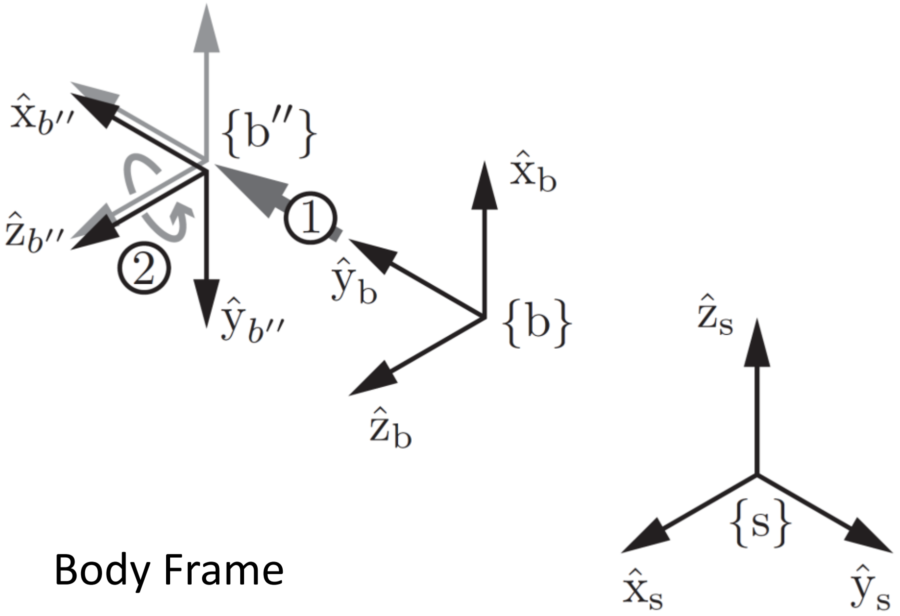
Fixed Frame

$$\hat{\omega} = (0, 0, 1) \quad \theta = 90^\circ \quad p = (0, 2, 0)$$

Displacing a Vector or a Frame in Body Frame

$$T_{sb''} = T_{sb}T = T_{sb}\,\mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega},\theta) \qquad \text{(body frame)}$$

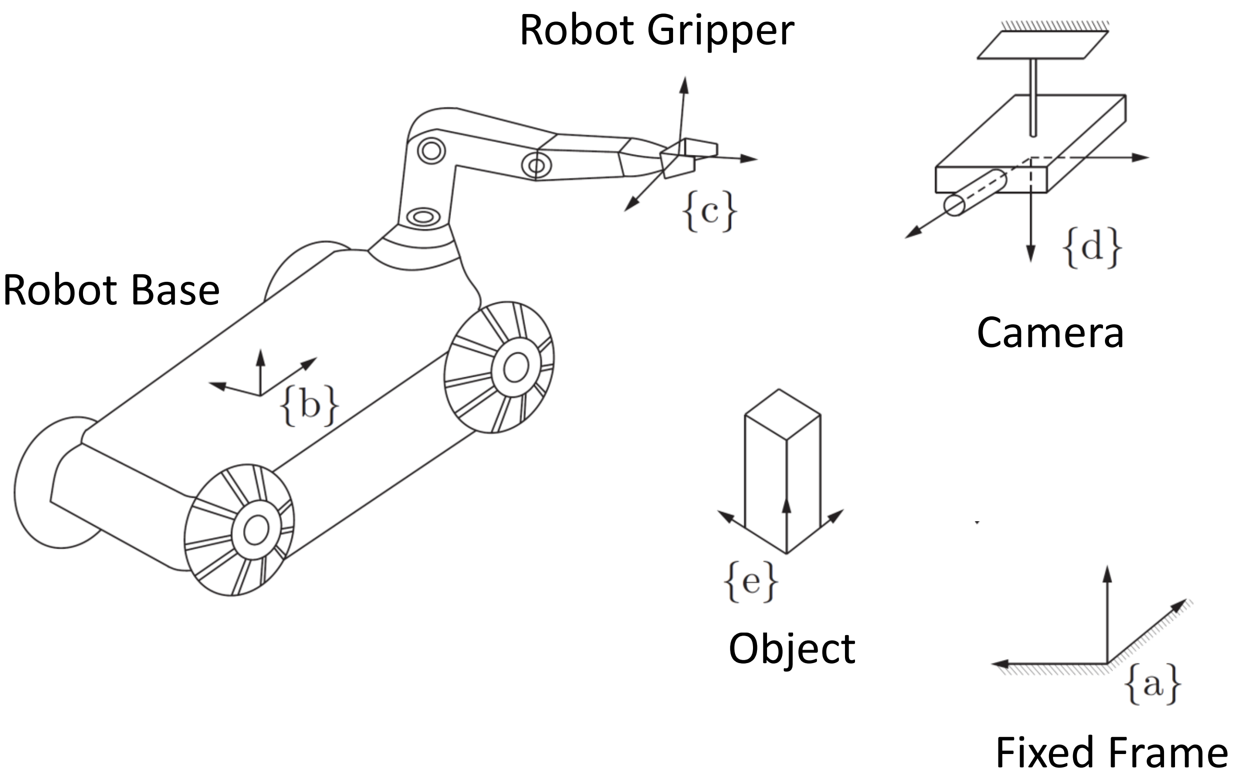
$$= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \\ 0 & 1 \end{bmatrix}$$

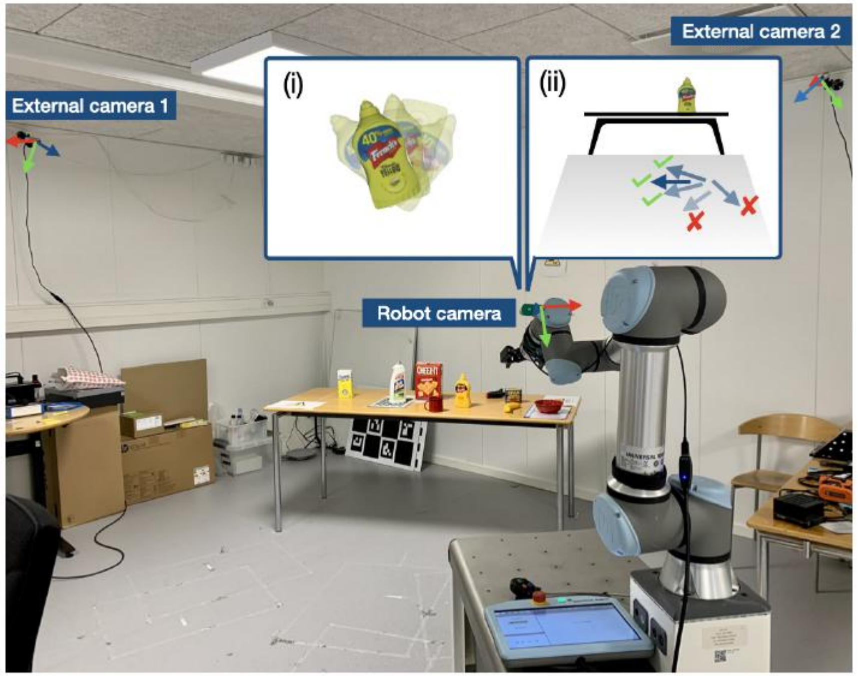


$$\hat{\omega} = (0,0,1) \quad \theta = 90^\circ \quad p = (0,2,0)$$

Body Frame

Transformation Matrices in Robotics



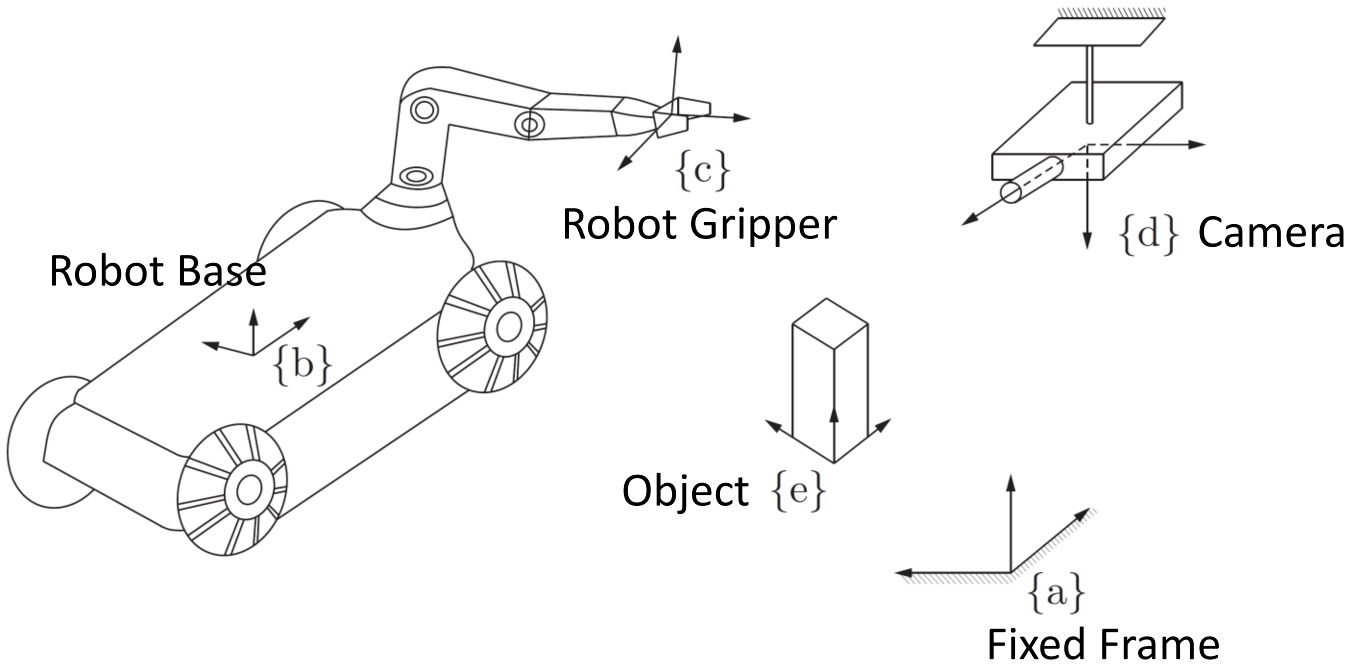


Multi-view object pose distribution tracking for pre-grasp planning on mobile robots. Naik et al., ICRA, 2022.

- How to move the robot arm to pick up the object?
- What information we need to know?

T_{ce} Pose of object in gripper

Transformation Matrices



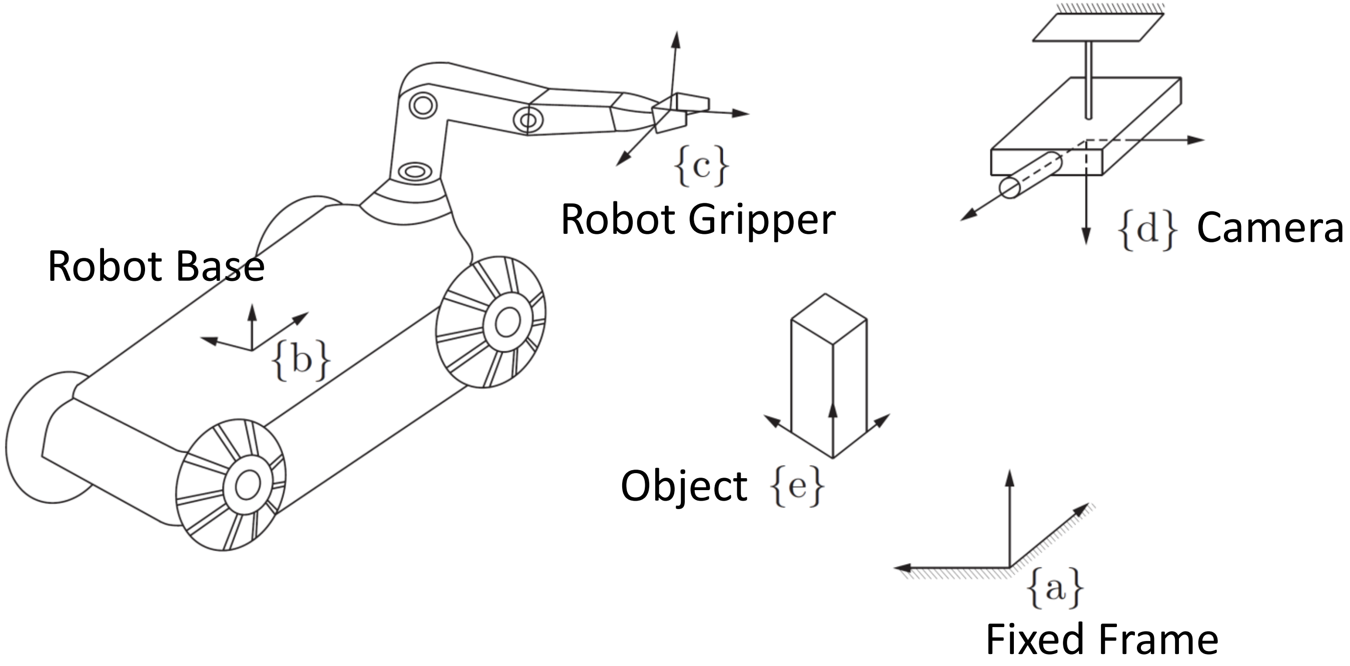
- Robot tracking in camera

https://research.nvidia.com/publication/2020-05_camera-robot-pose-estimation-single-image

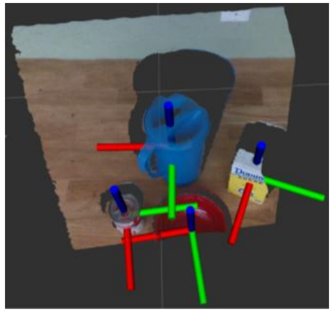
Robot in camera

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



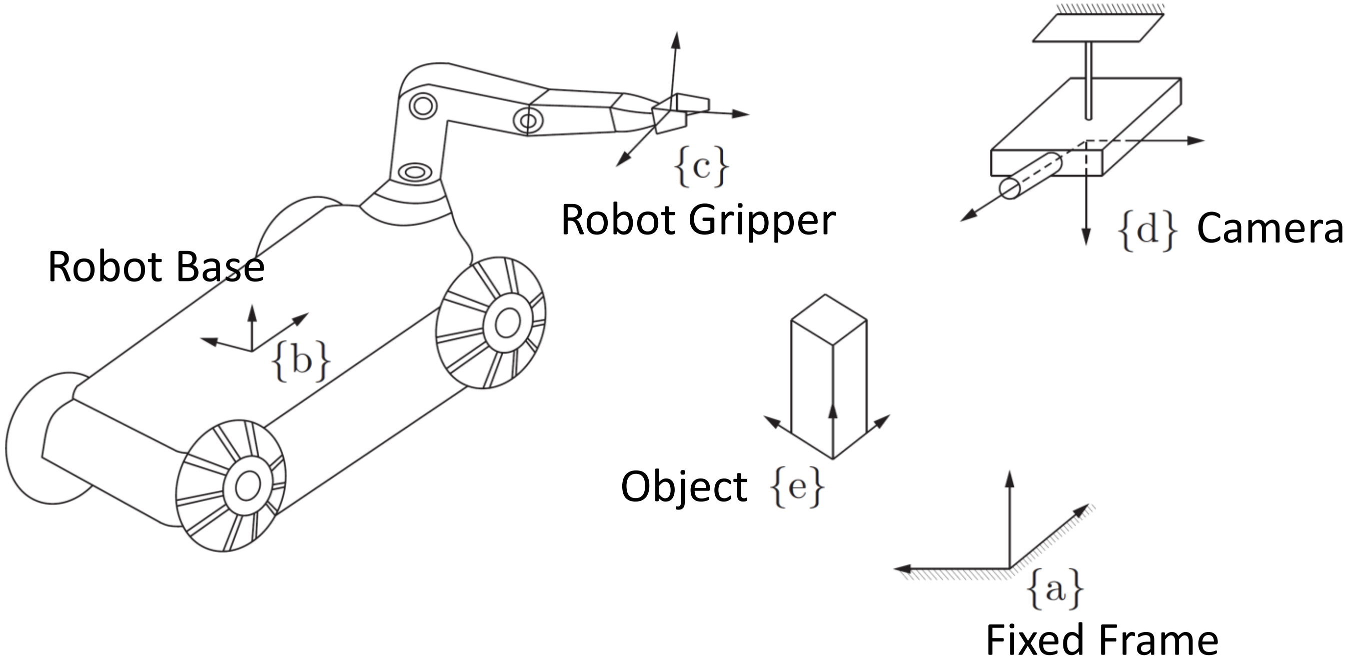
- Object pose estimation from camera



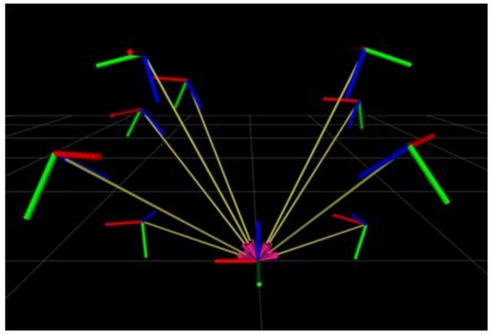
Object in camera

$$T_{de} = \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



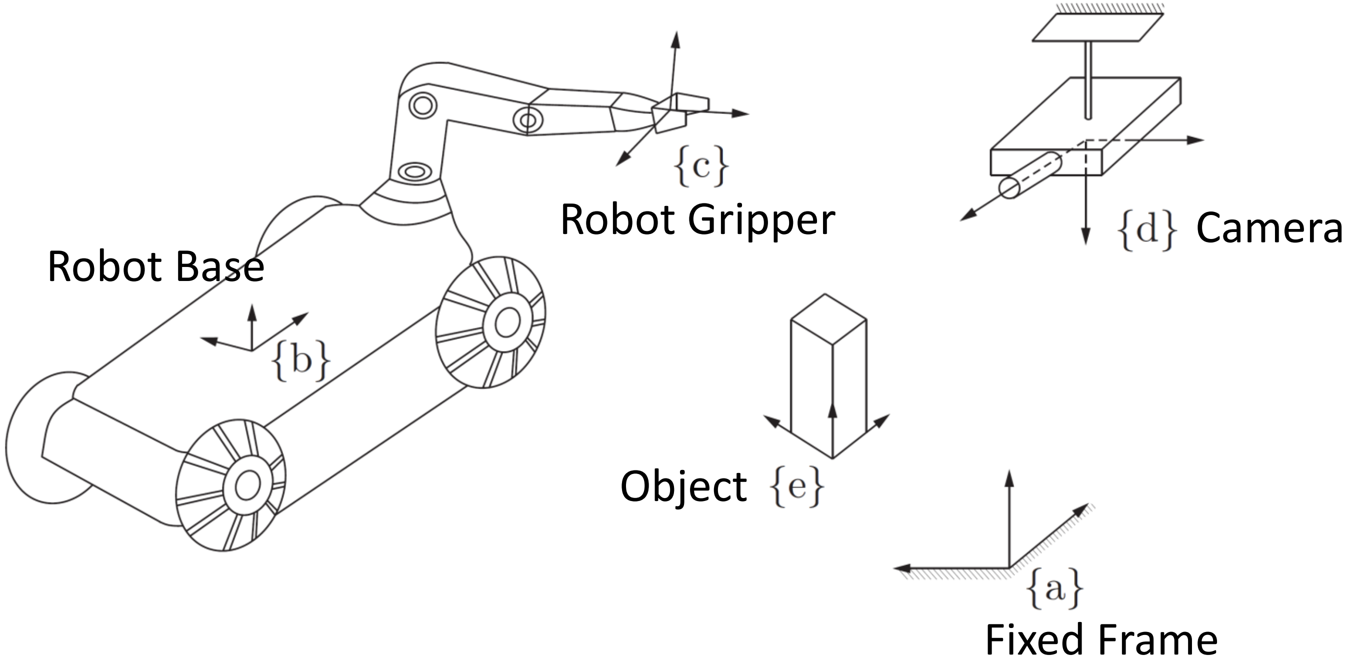
- Camera calibration



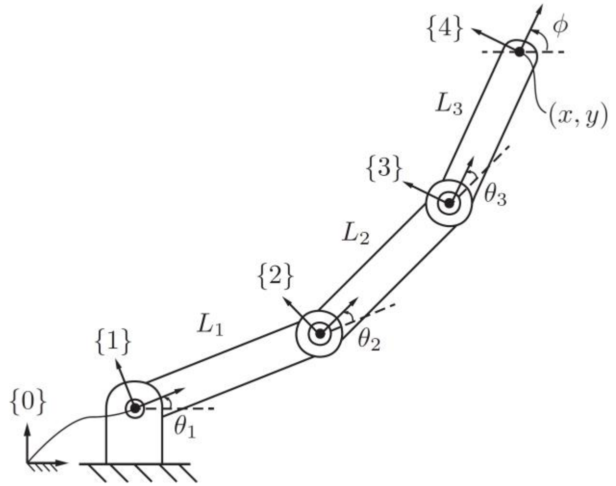
Camera in fixed frame

$$T_{ad} = \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



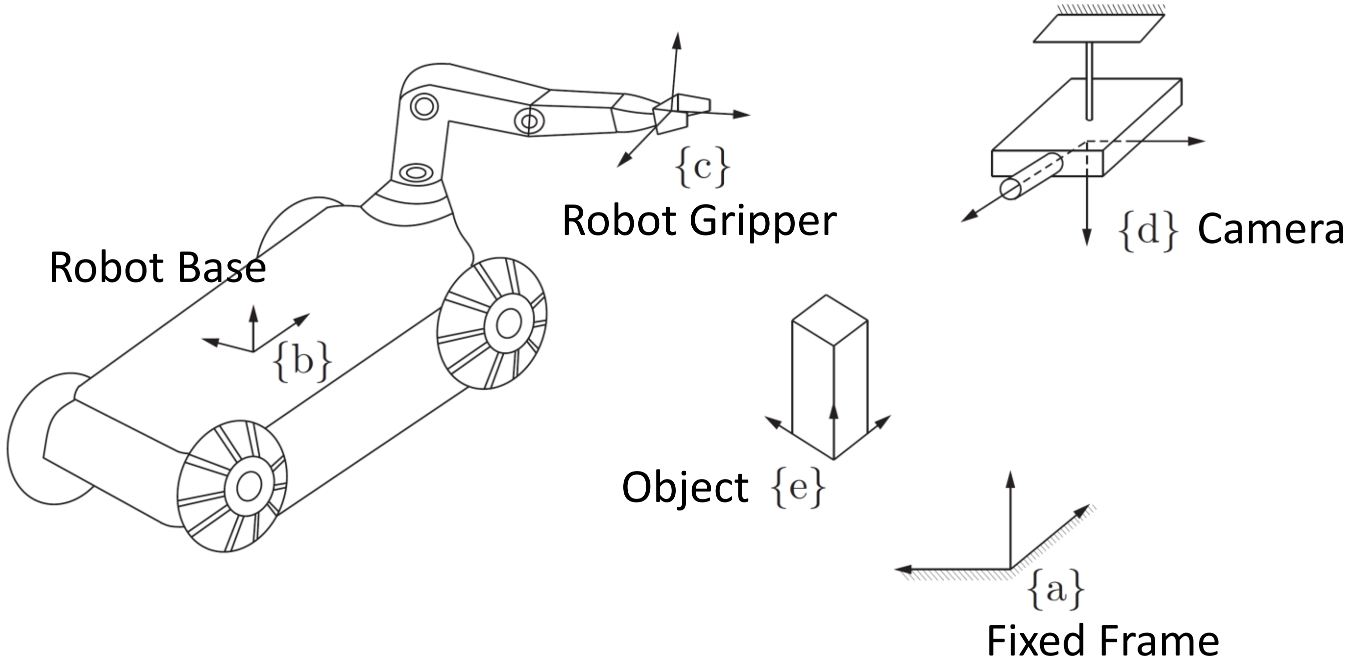
- Forward kinematics (future lectures)



Gripper in robot

$$T_{bc} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 30 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



- We know the following transformations

Robot in camera

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Object in camera

$$T_{de} = \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

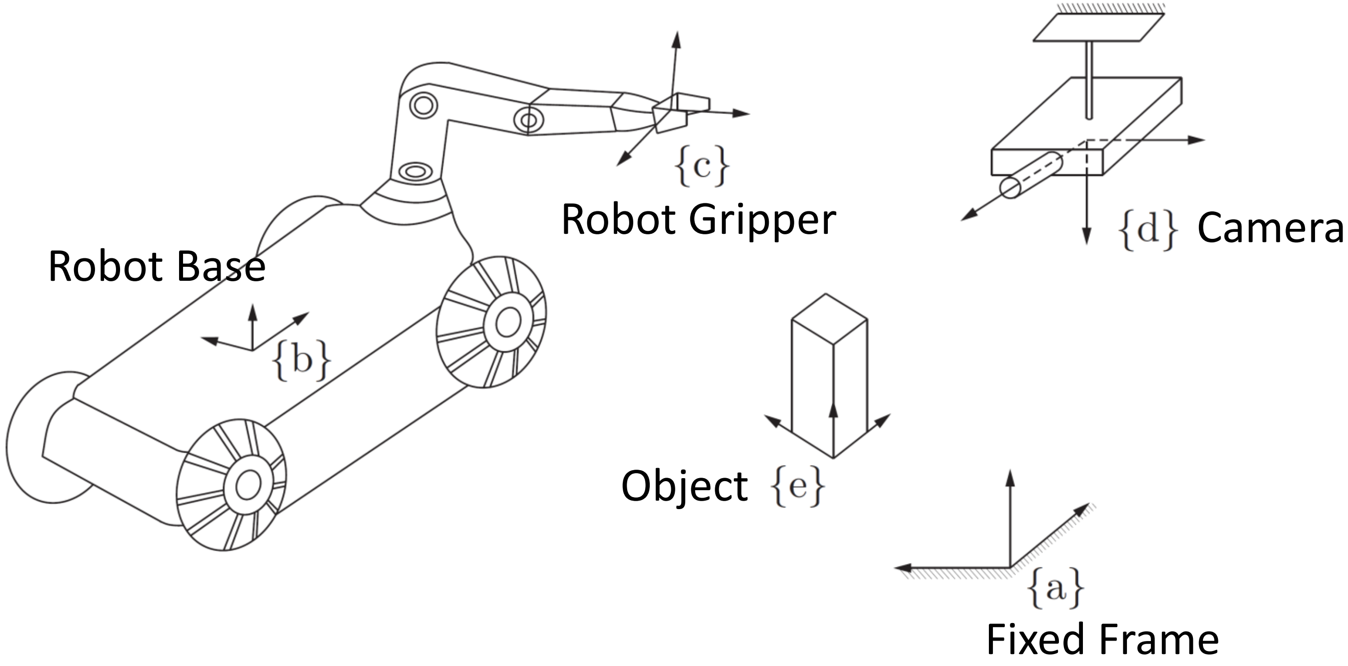
Camera in fixed frame

$$T_{ad} = \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Gripper in robot

$$T_{bc} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 30 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



- How to move the robot arm to pick up the object?

$$T_{ce}$$

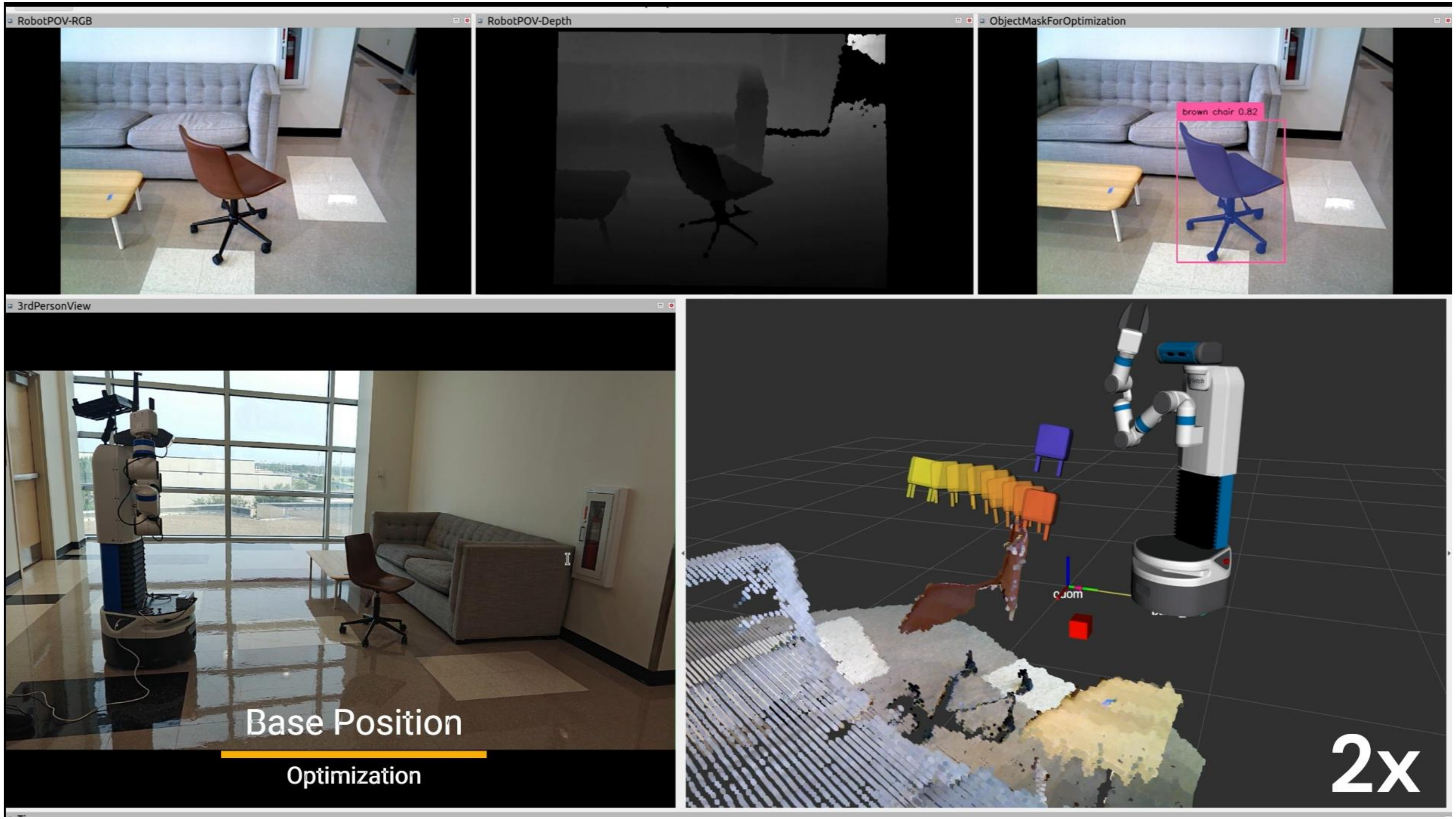
- We know T_{db} T_{de} T_{bc} T_{ad}

$$T_{ce} = T_{bc}^{-1} T_{db}^{-1} T_{de}$$

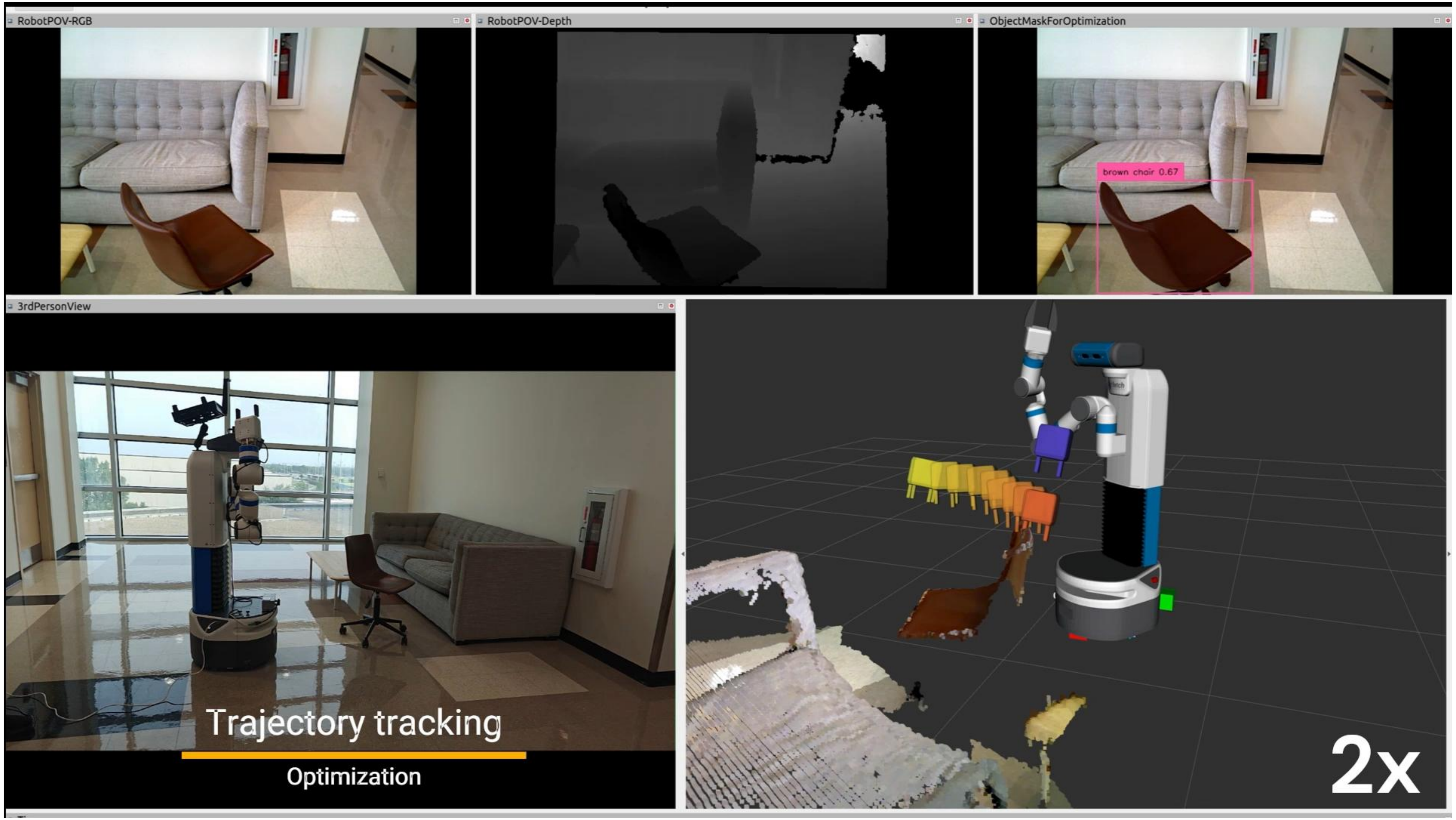
$$T_{ce} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -260/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 130/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Demo

Optimizing the Robot Base Location



Optimizing the Robot Trajectory



Summary

- Homogenous transformation matrices

- Uses of Transformation Matrices

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017