The Physics of Virtual Worlds

CS 6334 Virtual Reality

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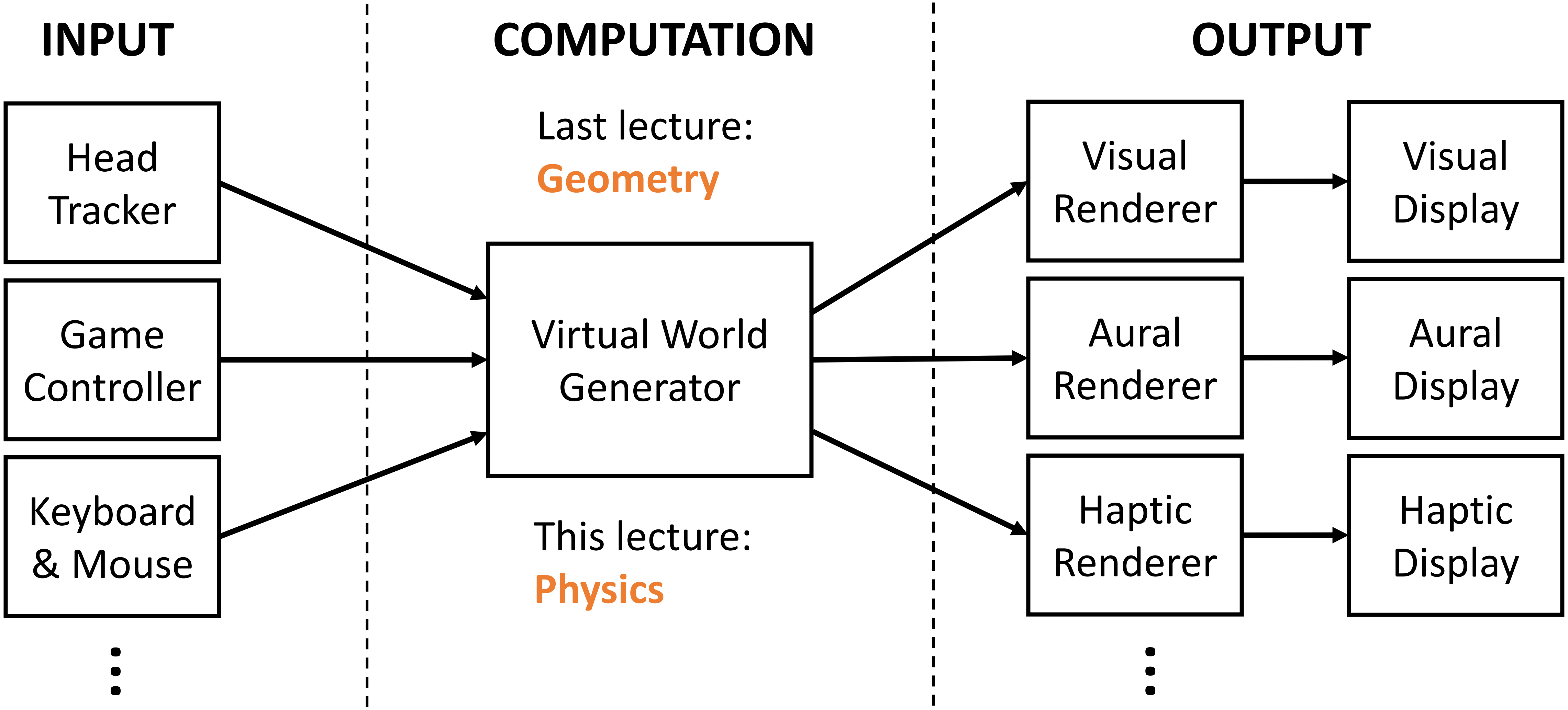
The University of Texas at Dallas

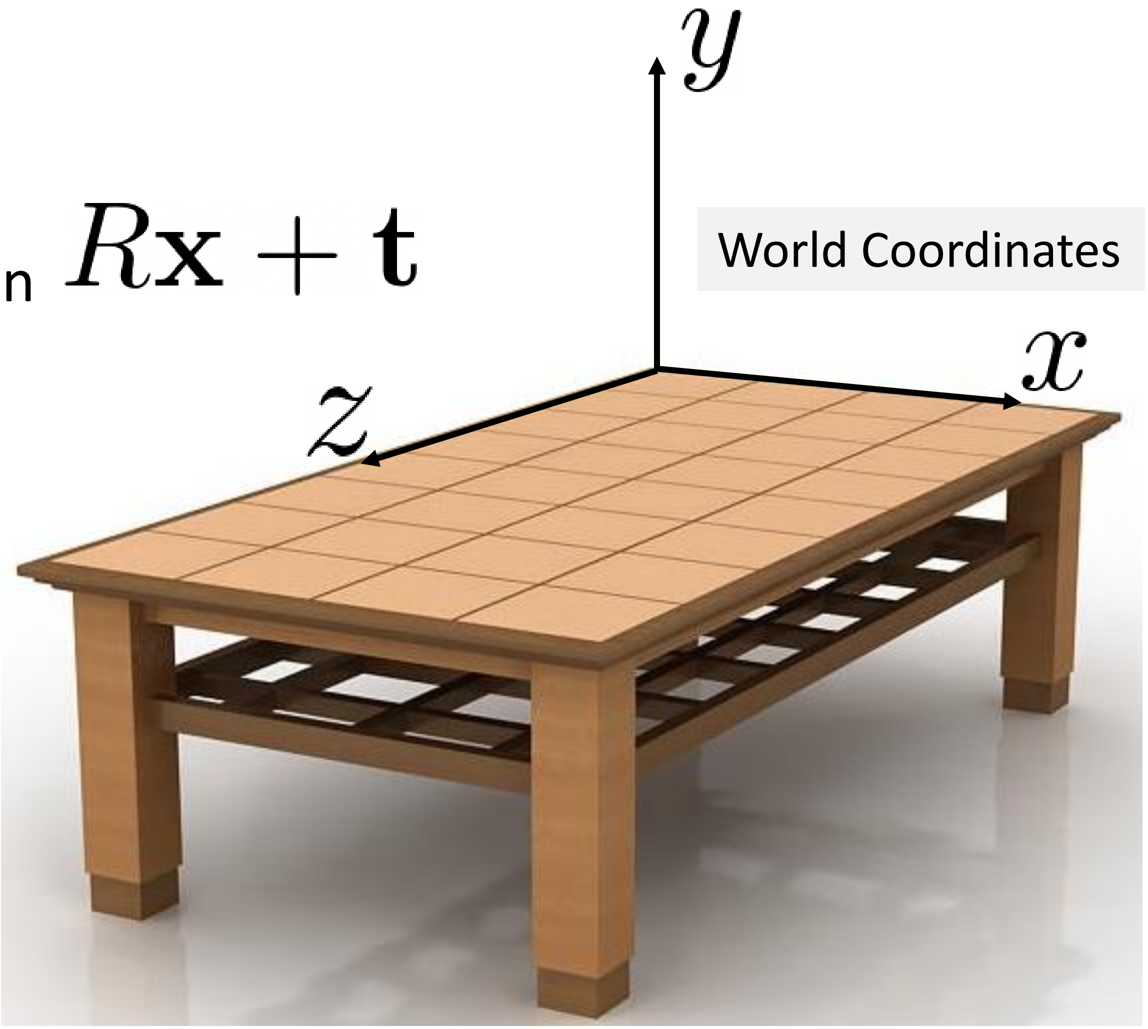
Review of VR Systems



The Geometry of Virtual Worlds

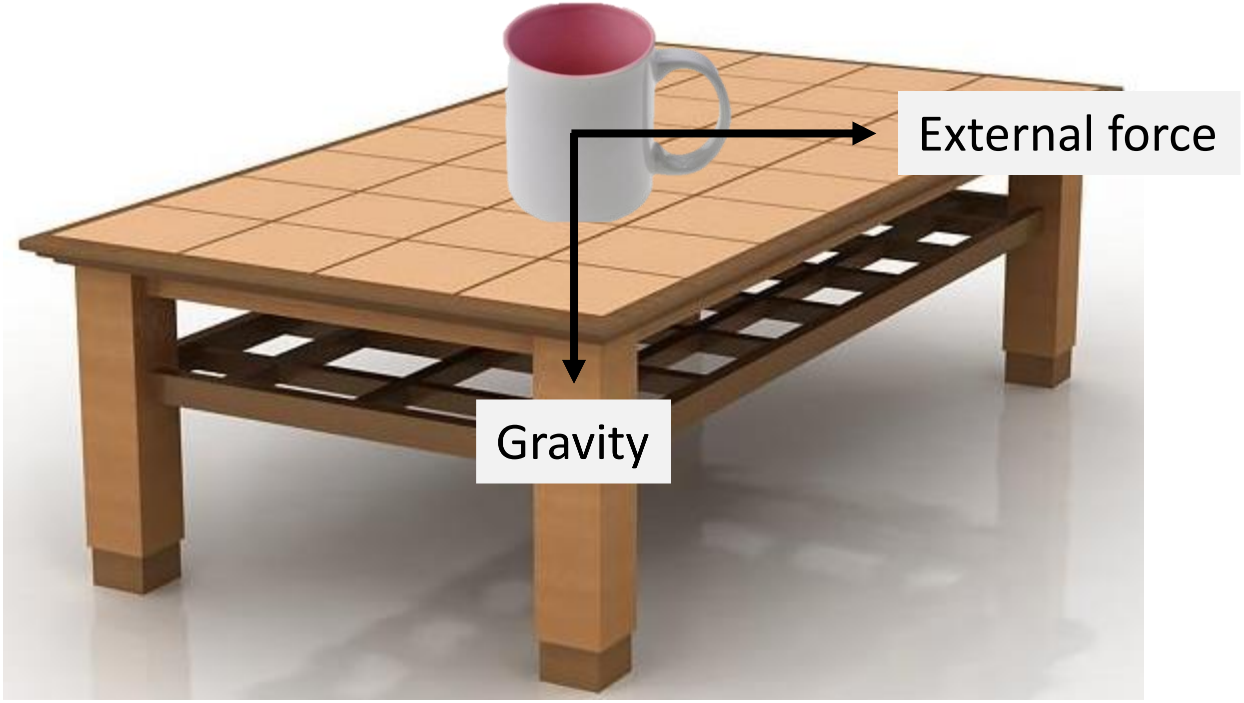
3D Rotation
3D Translation $R\mathbf{x} + \mathbf{t}$

World Coordinates



Object Coordinates

The Physics of Virtual Worlds



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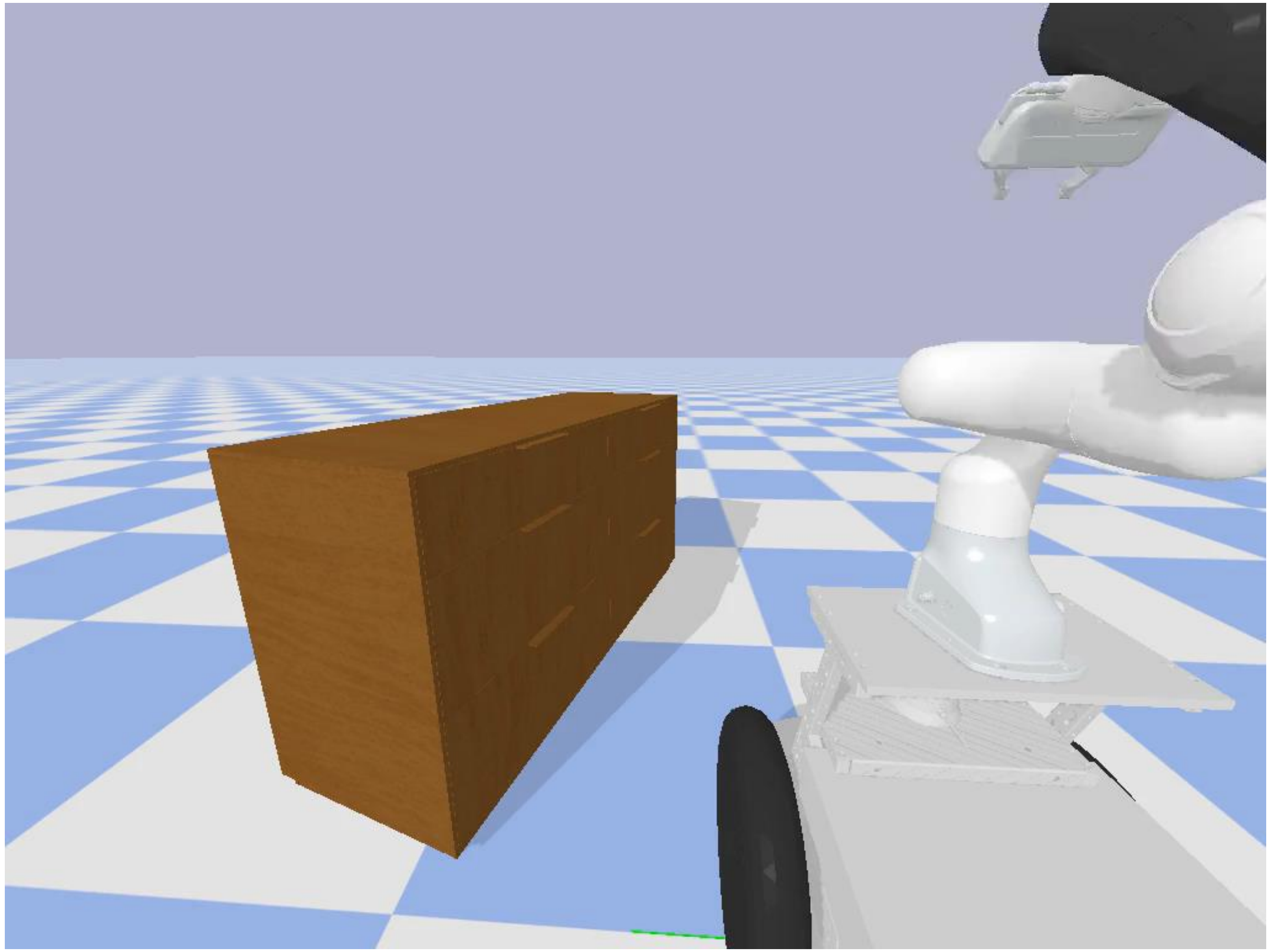
PyBullet Example

Programming HW1

PyBullet Example



Credit: Xiangyun Meng at UW

Physics Simulation

- Dynamical system

State of the virtual world
- Object positions
- Object shapes
- Forces
- Energy

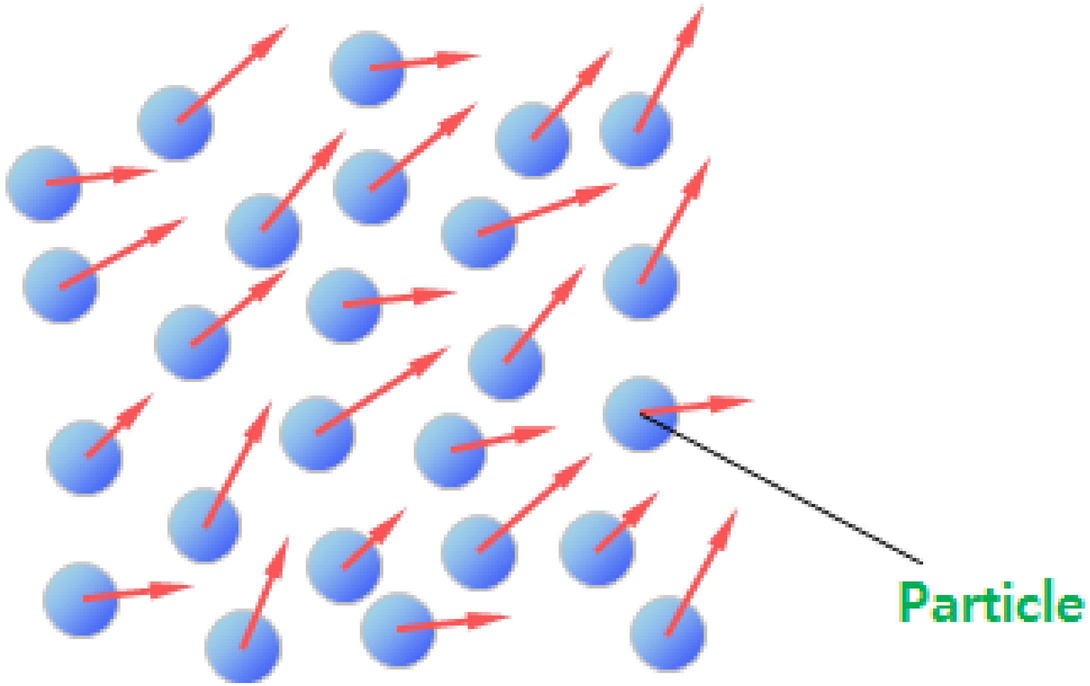
…

$$\mathbf{s}_t \longrightarrow \boxed{\text{Physics Engine}} \longrightarrow \mathbf{s}_{t+1}$$

Pendulum

Particle Dynamics

- Determine the states of particles (e.g., position)



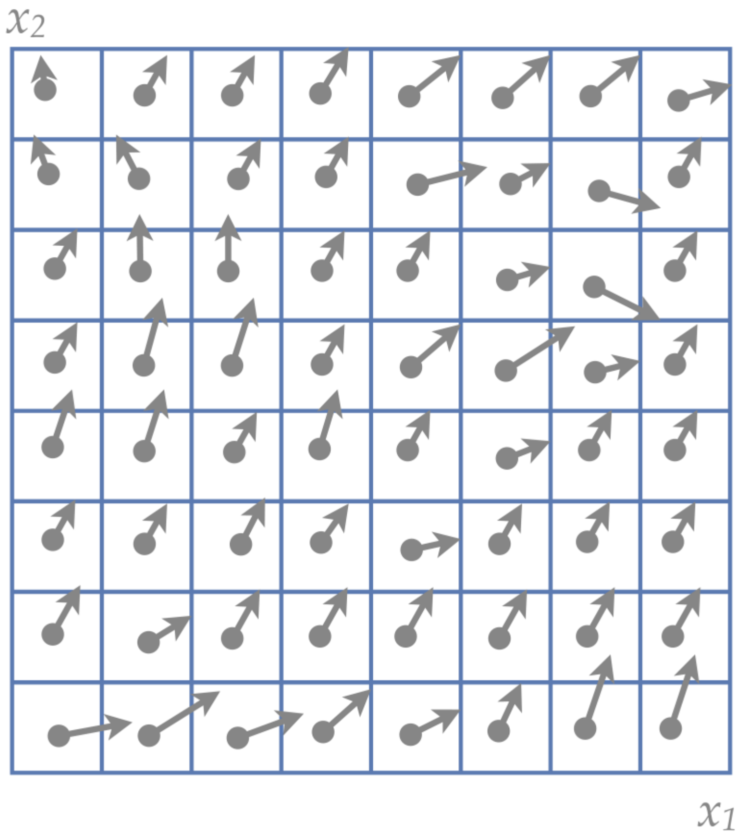
Particle Dynamics

- Determine the position of a mass-less particle
- Given velocity field $\mathbf{v}(\mathbf{x}, t)$
- Initial Value Problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\frac{d\mathbf{x}_p(t)}{dt} = \dot{\mathbf{x}}_p(t) = \mathbf{v}(\mathbf{x}_p, t)$$

How to calculate $\mathbf{x}_p(t)$



Differential Equations

- A differential equation is an equation that relates one or more functions and their derivatives

$$\frac{d\mathbf{x}_p(t)}{dt} = \dot{\mathbf{x}}_p(t) = \mathbf{v}(\mathbf{x}_p, t)$$

- Ordinary Differential Equation (ODE)
  - An equation that contains functions of only one independent variable and its derivatives
  - First-order ODE

Initial Value Problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

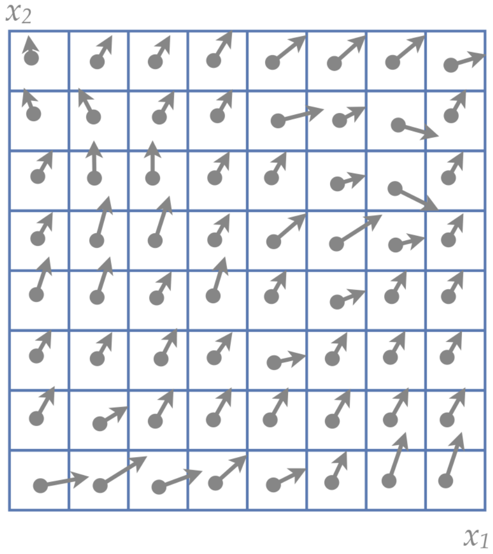
$$\frac{d\mathbf{x}_p(t)}{dt} = \dot{\mathbf{x}}_p(t) = \mathbf{v}(\mathbf{x}_p, t)$$

- Euler integration

$$\frac{d\mathbf{x}_p(t)}{dt} = \lim_{\epsilon \to 0} \frac{\mathbf{x}_p(t+\epsilon) - \mathbf{x}_p(t)}{\epsilon}$$

$$\frac{d\mathbf{x}_p(t)}{dt} \approx \frac{\mathbf{x}_p(t+\Delta t) - \mathbf{x}_p(t)}{\Delta t}$$

$$\frac{\mathbf{x}_p(t+\Delta t) - \mathbf{x}_p(t)}{\Delta t} = \mathbf{v}(\mathbf{x}_p, t)$$



Position of the mass-less particle

$$\mathbf{x}_p(t+\Delta t) = \mathbf{x}_p(t) + \Delta t \cdot \mathbf{v}(\mathbf{x}_p, t)$$

Particle Dynamics

- Determine the position of a particle with mass
- Newton's second law

$$\mathbf{f} = m\mathbf{a}$$

Vector sum of all forces applied to each body in a system, newtons (N)

Mass of the body, kg

Vector acceleration of each body with respect to an inertial reference frame, m/sec^2

Acceleration of gravity g=9.81 m/sec^2

Momentum

- The momentum of a body is

$$\mathbf{p}(t) = m\mathbf{v}(t)$$

Mass of the body, kg

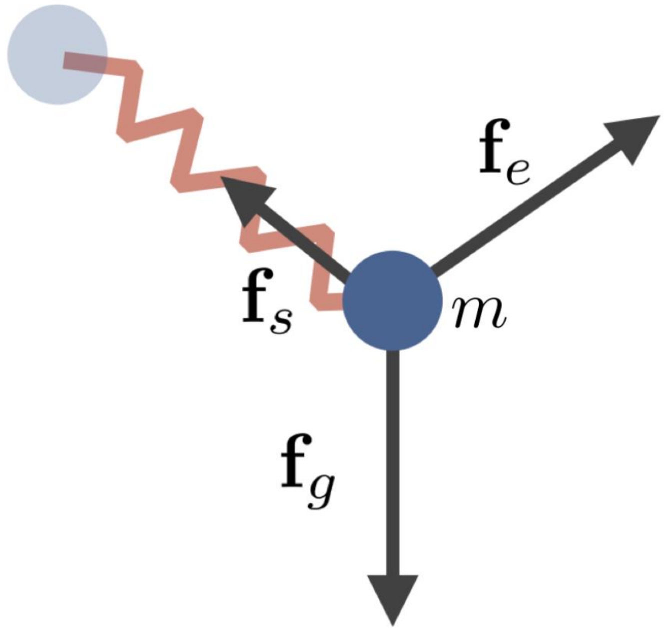
Velocity of the body, m/sec

- Newton's second law

$$\mathbf{f}(t) = \frac{d}{dt}\mathbf{p}(t) = m\frac{d}{dt}\mathbf{v}(t) = m\mathbf{a}(t)$$

Newton's Second Law

- Example



$$ma = \mathbf{f}_s + \mathbf{f}_g + \mathbf{f}_e$$

acceleration → velocity → position

Bargteil, A., Shinar T. An introduction to physics-based animation, ACM SIGGRAPH 2018 Courses, 2018

A Particle with Mass

- Initial value problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\frac{d^2\mathbf{x}_p(t)}{dt^2} = \ddot{\mathbf{x}}_p(t) = \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

- First-order equations

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\mathbf{v}_p(0) = \mathbf{v}_0$$

$$\frac{d\mathbf{x}_p(t)}{dt} = \dot{\mathbf{x}}_p(t) = \mathbf{v}_p(t)$$

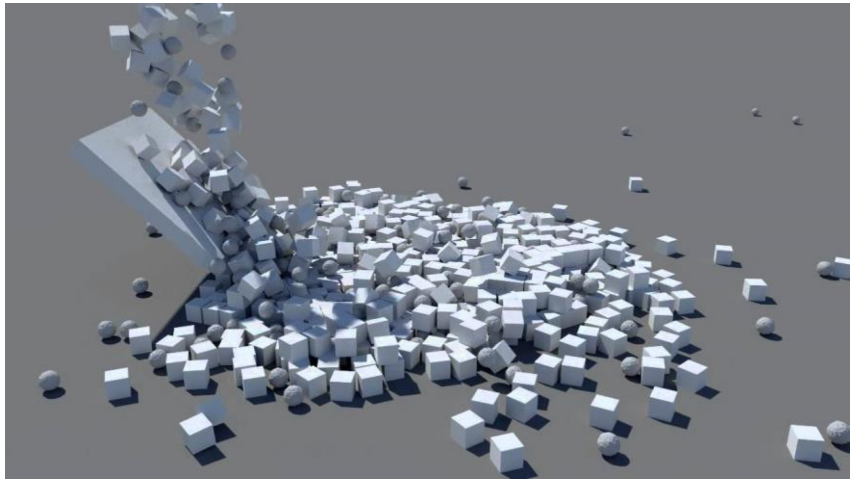
$$\frac{d\mathbf{v}_p(t)}{dt} = \dot{\mathbf{v}}_p(t) = \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

Euler's method

$$\mathbf{v}_p(t + \Delta t) = \mathbf{v}_p(t) + \Delta t \cdot \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

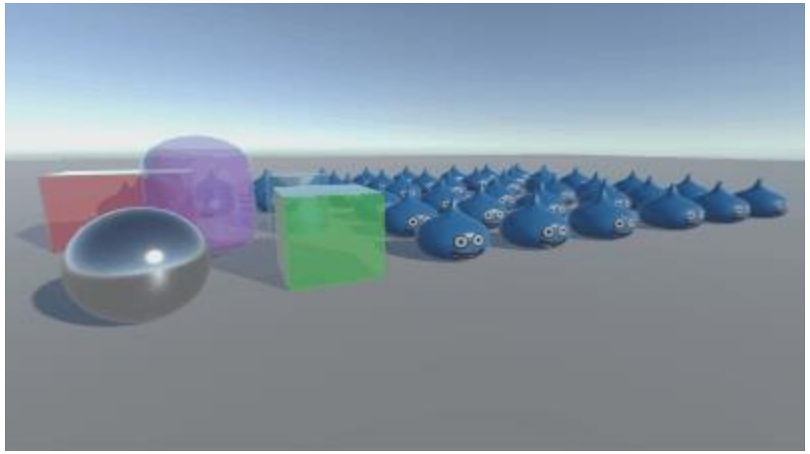
$$\mathbf{x}_p(t + \Delta t) = \mathbf{x}_p(t) + \Delta t \cdot \mathbf{v}_p(t + \Delta t)$$

Materials



Rigid bodies

- No deformation



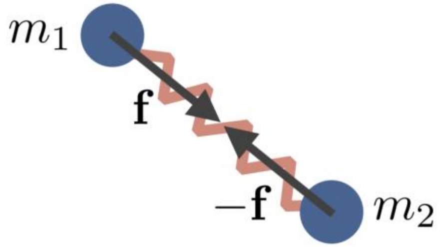
Soft bodies

- Deform elastically and plastically

Fluids

- Air, water, honey, etc.

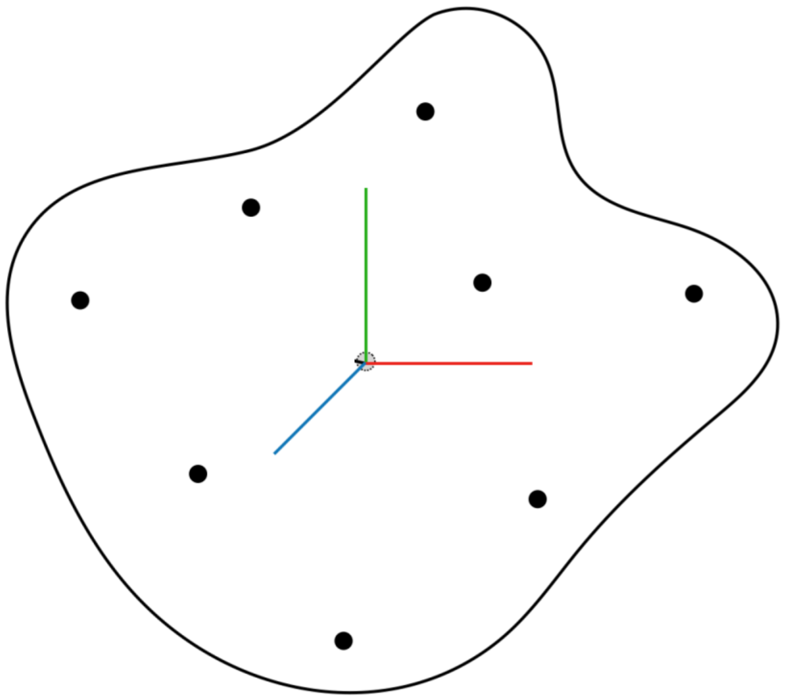
Particles with springs



Rigid Bodies

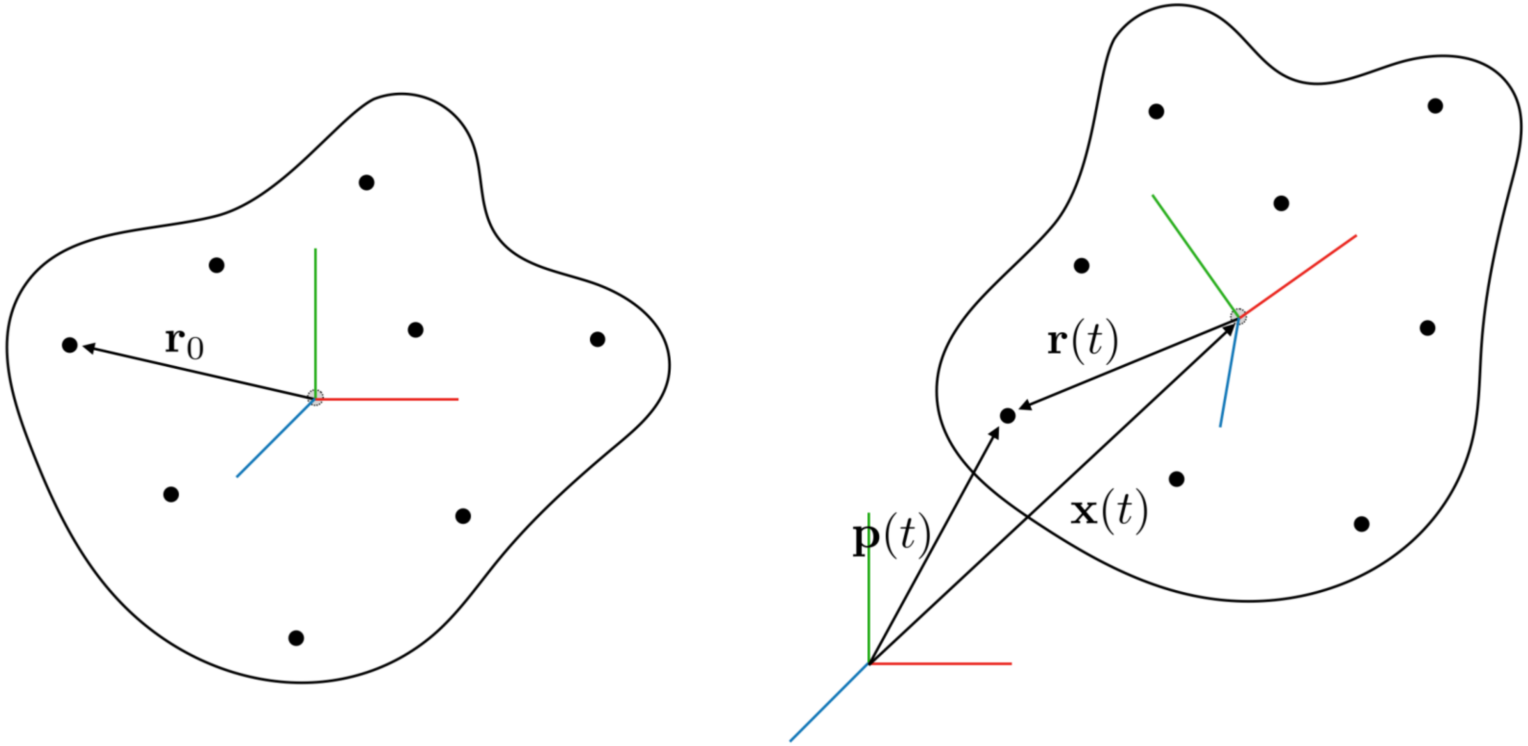
- No deformation
- 6 DOF: 3D translation and 3D rotation
- Particles with very stiff springs
- Center of mass

$$\mathbf{x}_{com} = \frac{\sum_{i=1}^{N} m_i \mathbf{p}_i}{\sum_{i=1}^{N} m_i}$$



Bargteil, A., Shinar T. [An introduction to physics-based animation](), ACM SIGGRAPH 2018 Courses, 2018

Object Space vs. World Space



(a) *Object space.* **(b)** *World space.*

World position

$$\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_0$$

$$\mathbf{p}(t) = \mathbf{x}(t) + \mathbf{r}(t)$$

$$\mathbf{p}(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_0$$

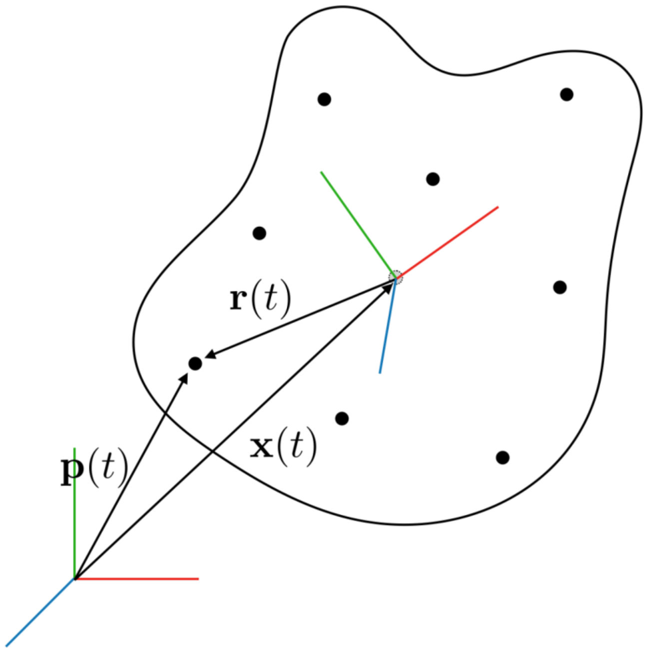
Linear Velocity

$$\mathbf{p}(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_0$$

$$\mathbf{v}(t) = \dot{\mathbf{p}}(t) = \dot{\mathbf{x}}(t) + \dot{\mathbf{R}}(t)\mathbf{r}_0$$

Linear velocity

- Motion of the particle due to linear velocity of the body



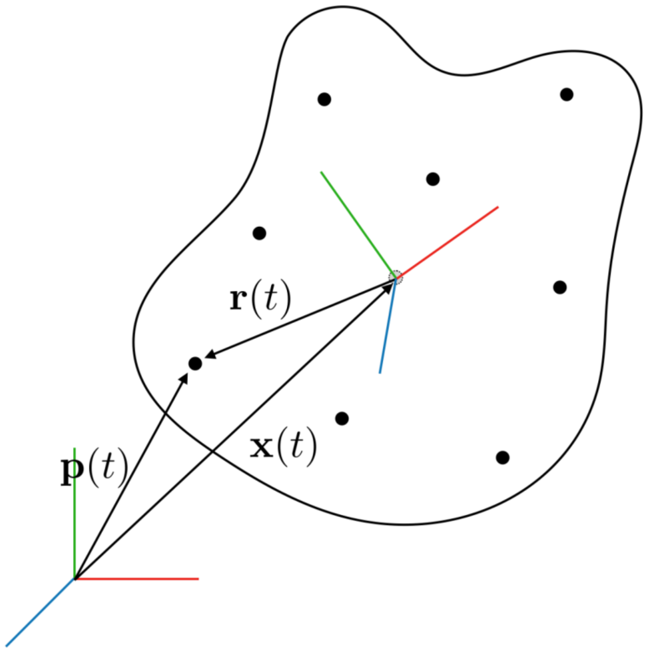
(b) *World space.*

Instantaneous Rotation

$$\mathbf{p}(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_0$$

$$\mathbf{v}(t) = \dot{\mathbf{p}}(t) = \dot{\mathbf{x}}(t) + \dot{\mathbf{R}}(t)\mathbf{r}_0$$

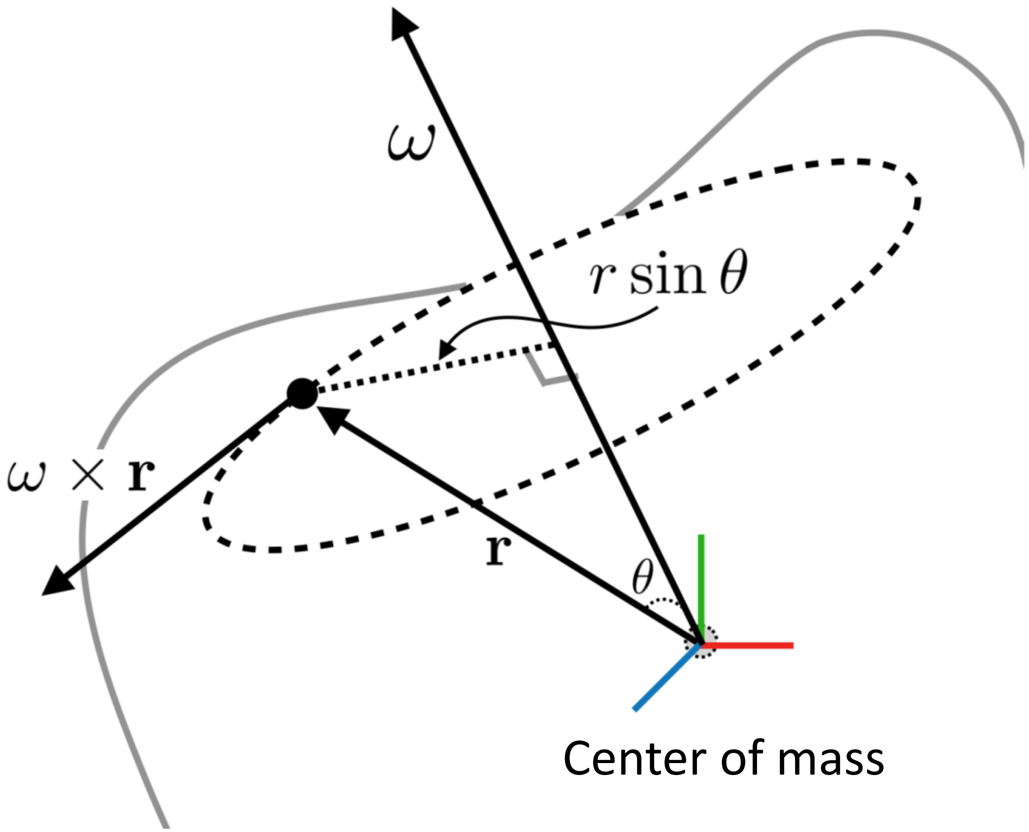
Motion of the particle due to the instantaneous rotation of the body about its center of mass



(b) *World space.*

Angular Velocity ω

Euler's rotation theorem $\dot{\mathbf{R}}(t)\mathbf{r}_0$



Center of mass

ω
- The vector whose direction is the instantaneous axis of rotation
- Length is the rate of rotation in radians per second

$$\dot{\mathbf{R}}(t)\mathbf{r}_0 = \boldsymbol{\omega}(t) \times \mathbf{r}(t)$$

$$\mathbf{v}(t) = \dot{\mathbf{p}}(t) = \dot{\mathbf{x}}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}(t)$$

$$\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_0 \qquad \dot{\mathbf{R}}(t) = \boldsymbol{\omega}(t) \times \mathbf{R}(t)$$

Linear Momentum

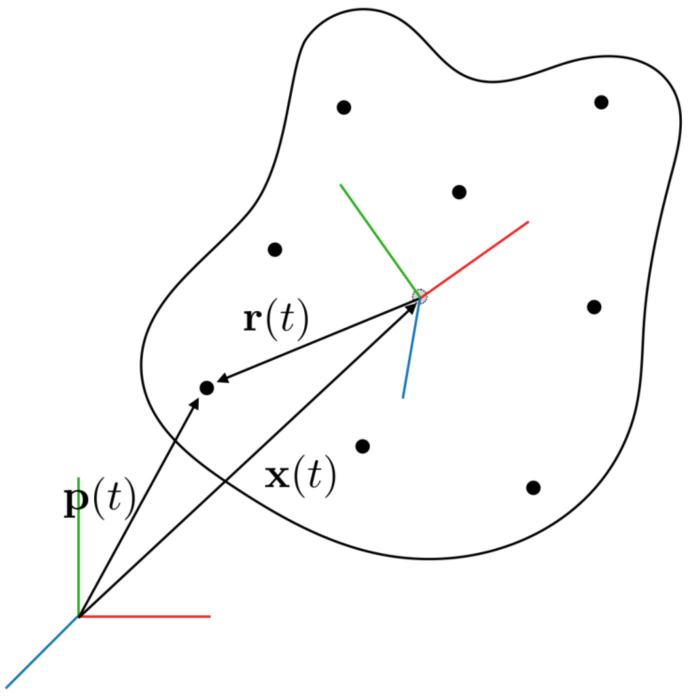
$$\mathbf{P}(t) = \sum_{i=1}^{N} m_i \mathbf{v}_i(t)$$

$$\mathbf{P}(t) = \sum_{i=1}^{N} m_i \left(\dot{\mathbf{x}}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}_i(t)\right)$$

$$= \sum_{i=1}^{N} m_i \dot{\mathbf{x}}(t) + \boldsymbol{\omega}(t) \times \left(\underbrace{\sum_{i=1}^{N} m_i \mathbf{r}_i(t)}_{0}\right)$$

Derivation HW1

$$\mathbf{P}(t) = M\dot{\mathbf{x}}(t) \qquad M = \sum_{i=1}^{N} m_i$$



(b) *World space.*

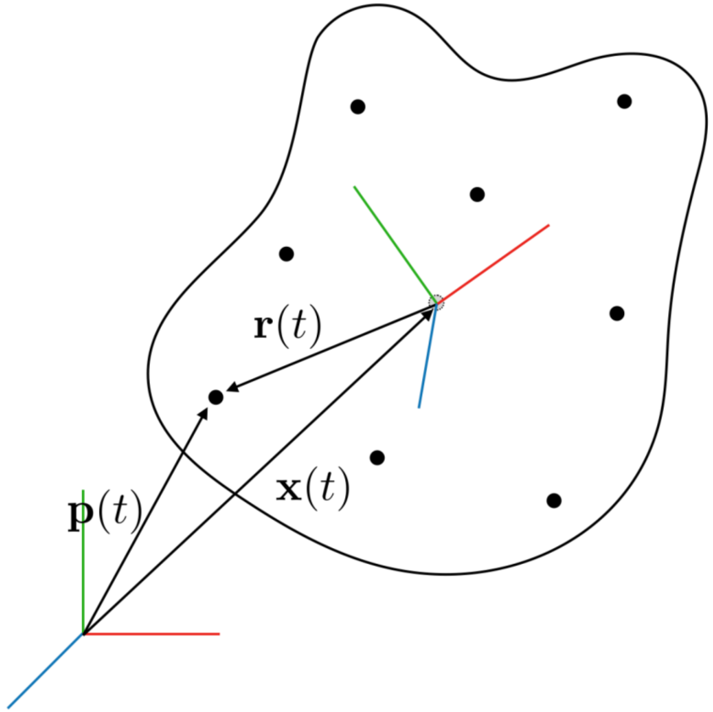
Angular Momentum

$$\mathbf{L}(t) = \sum_{i=1}^{N} \mathbf{r}_i(t) \times m_i \mathbf{v}_i(t)$$

$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times (\dot{\mathbf{x}}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}_i(t))$$

$$= \underbrace{\sum_{i=1}^{N} m_i \mathbf{r}_i(t)}_{0} \times \dot{\mathbf{x}}(t) + \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times \boldsymbol{\omega}(t) \times \mathbf{r}_i(t)$$

$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times (\boldsymbol{\omega}(t) \times \mathbf{r}_i(t))$$



(b) *World space.*

Angular Momentum

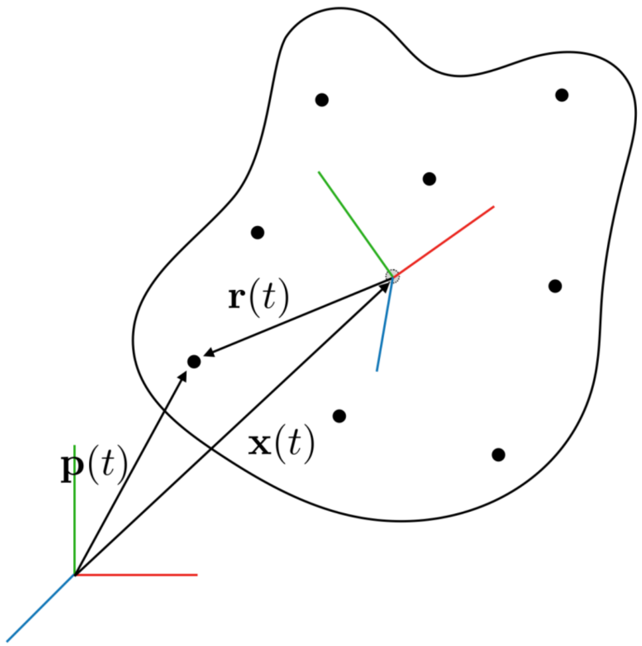
$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times (\boldsymbol{\omega}(t) \times \mathbf{r}_i(t))$$

$$\boldsymbol{\omega} \times \mathbf{r} = -\mathbf{r} \times \boldsymbol{\omega}$$

$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times (-\mathbf{r}_i(t) \times \boldsymbol{\omega}(t))$$

Cross product matrix $-\mathbf{r}^\star = \mathbf{r}^{\star T}$

$$\mathbf{r}^\star = \begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix}$$



(b) *World space.*

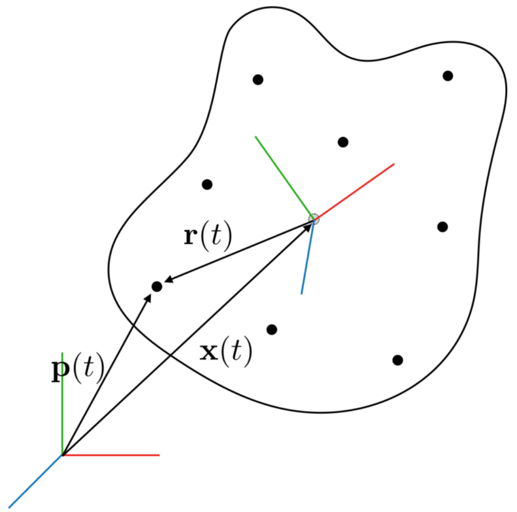
$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)(\mathbf{r}_i^{\star T}(t)\boldsymbol{\omega}(t))$$

$$= \left(\sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)\mathbf{r}_i^{\star T}(t)\right)\boldsymbol{\omega}(t)$$

Angular Momentum

$$\mathbf{L}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)(\mathbf{r}_i^{\star T}(t)\boldsymbol{\omega}(t))$$

$$= \left(\sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)\mathbf{r}_i^{\star T}(t) \right) \boldsymbol{\omega}(t)$$



(b) *World space.*

Inertia tensor

$$\mathbf{I}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)\mathbf{r}_i^{\star T}(t)$$

$$\mathbf{L}(t) = \mathbf{I}(t)\boldsymbol{\omega}(t)$$

$$\mathbf{I}(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^\star(t)\mathbf{r}_i^{\star T}(t)$$

$$= \sum_{i=1}^{N} m_i \left(\mathbf{r}_i^T \mathbf{r}_i \boldsymbol{\delta} - \mathbf{r}_i \mathbf{r}_i^T \right)$$

$$= \mathbf{R}(t) \sum_{i=1}^{N} m_i \left(\mathbf{r}_{0i}^T \mathbf{r}_{0i} \boldsymbol{\delta} - \mathbf{r}_{0i} \mathbf{r}_{0i}^T \right) \mathbf{R}(t)^T$$

$$= \mathbf{R}(t)\mathbf{I}_0\mathbf{R}(t)^T.$$

$$\mathbf{r}^\star \mathbf{r}^{\star T} = \mathbf{r}^T \mathbf{r} \boldsymbol{\delta} - \mathbf{r}\mathbf{r}^T$$

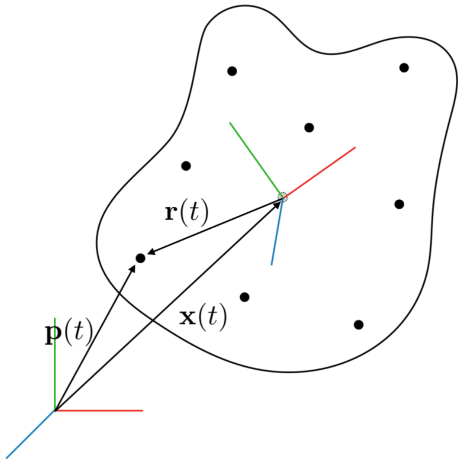
$\boldsymbol{\delta}$ is the 3 × 3 identity matrix

$\mathbf{r} = \mathbf{R}\mathbf{r}_0$

Force and Torque

Linear momentum $\mathbf{P}(t) = M\dot{\mathbf{x}}(t)$ $\quad M = \sum_{i=1}^{N} m_i$

Angular momentum $\mathbf{L}(t) = \mathbf{I}(t)\boldsymbol{\omega}(t)$



(b) *World space.*

- Newton's second law

$$\frac{d}{dt}\begin{pmatrix}\mathbf{P}(t)\\ \mathbf{L}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{f}(t)\\ \boldsymbol{\tau}(t)\end{pmatrix}$$

Force

Torque

- When a force apply to center of mass

$$\mathbf{a} = \mathbf{f}/M$$

- When a force apply to a point

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}$$

Dynamics of Rigid Bodies

$$\mathbf{v}(t) = \frac{\mathbf{P}(t)}{M} \qquad \mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_0\mathbf{R}(t)^T \qquad \boldsymbol{\omega}(t) = \mathbf{I}(t)^{-1}\mathbf{L}(t)$$

$$\frac{d}{dt}\begin{pmatrix} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ \boldsymbol{\omega}^{\star}(t)\mathbf{R}(t) \\ \mathbf{f}(t) \\ \boldsymbol{\tau}(t) \end{pmatrix}$$

Linear Velocity

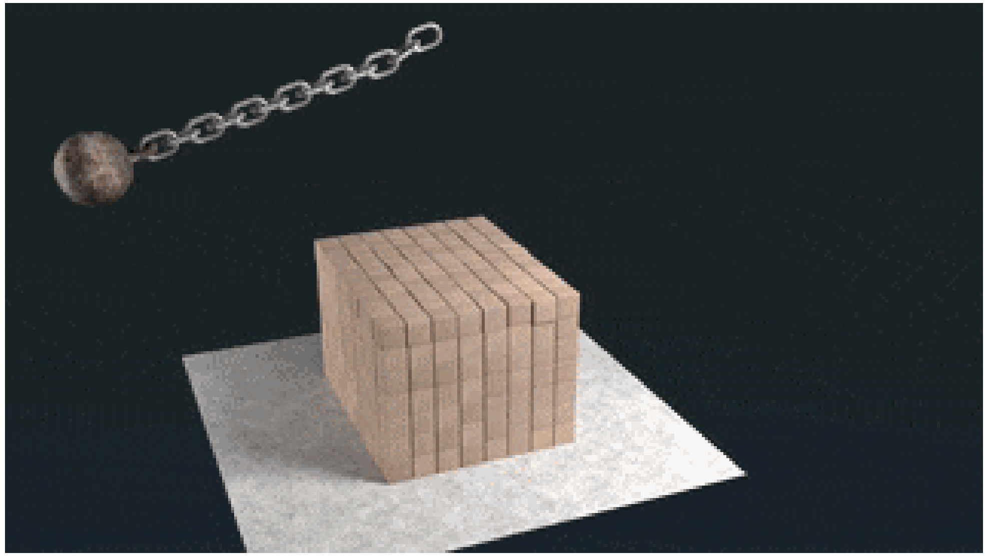
Angular Velocity

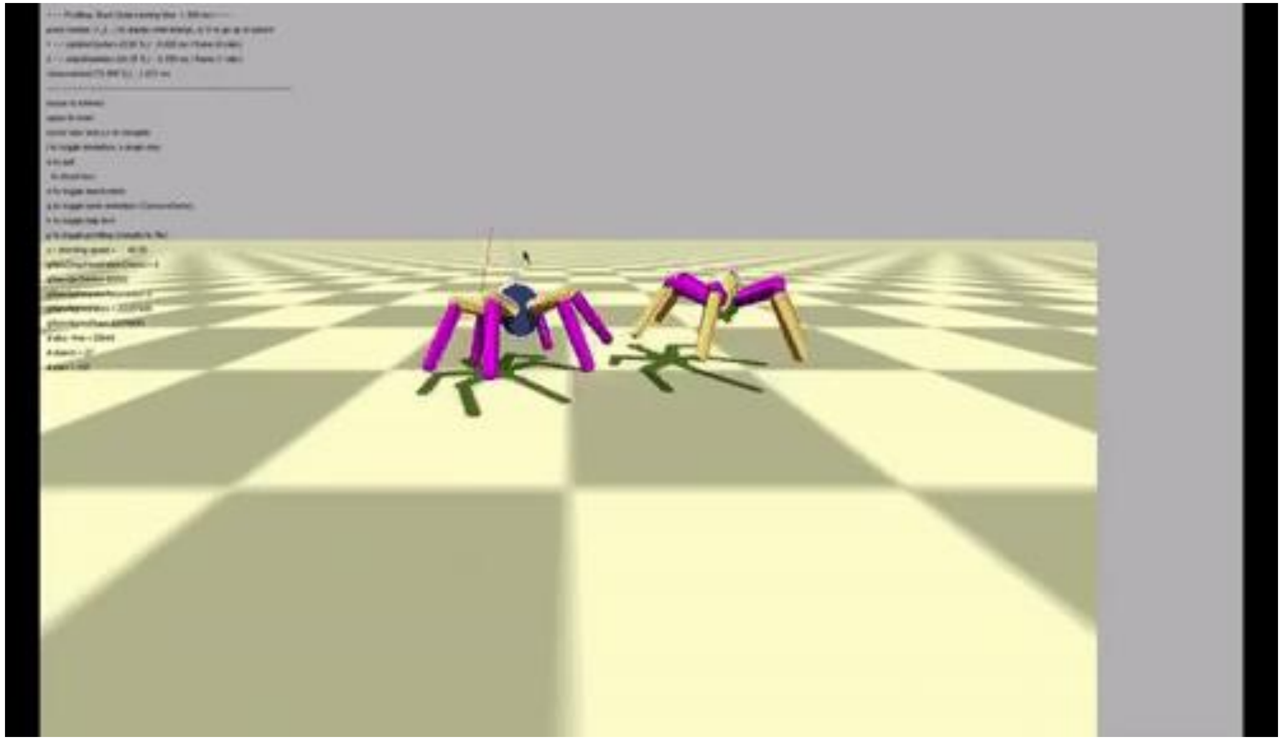
Force

Torque

Rigid Body Simulation Examples

$$\mathbf{s}_t \longrightarrow \boxed{\text{Physics Engine}} \longrightarrow \mathbf{s}_{t+1}$$





https://gfycat.com/

Further Readings

- Section 8.1, 8.3 in Virtual Reality, Steven LaValle
- Bargteil, A., Shinar T. An introduction to physics-based animation, ACM SIGGRAPH 2018 Courses, 2018.
- Rick Parent. Computer Animation: Algorithms and Techniques, 2012.