The Geometry of Virtual Worlds

CS 6334 Virtual Reality

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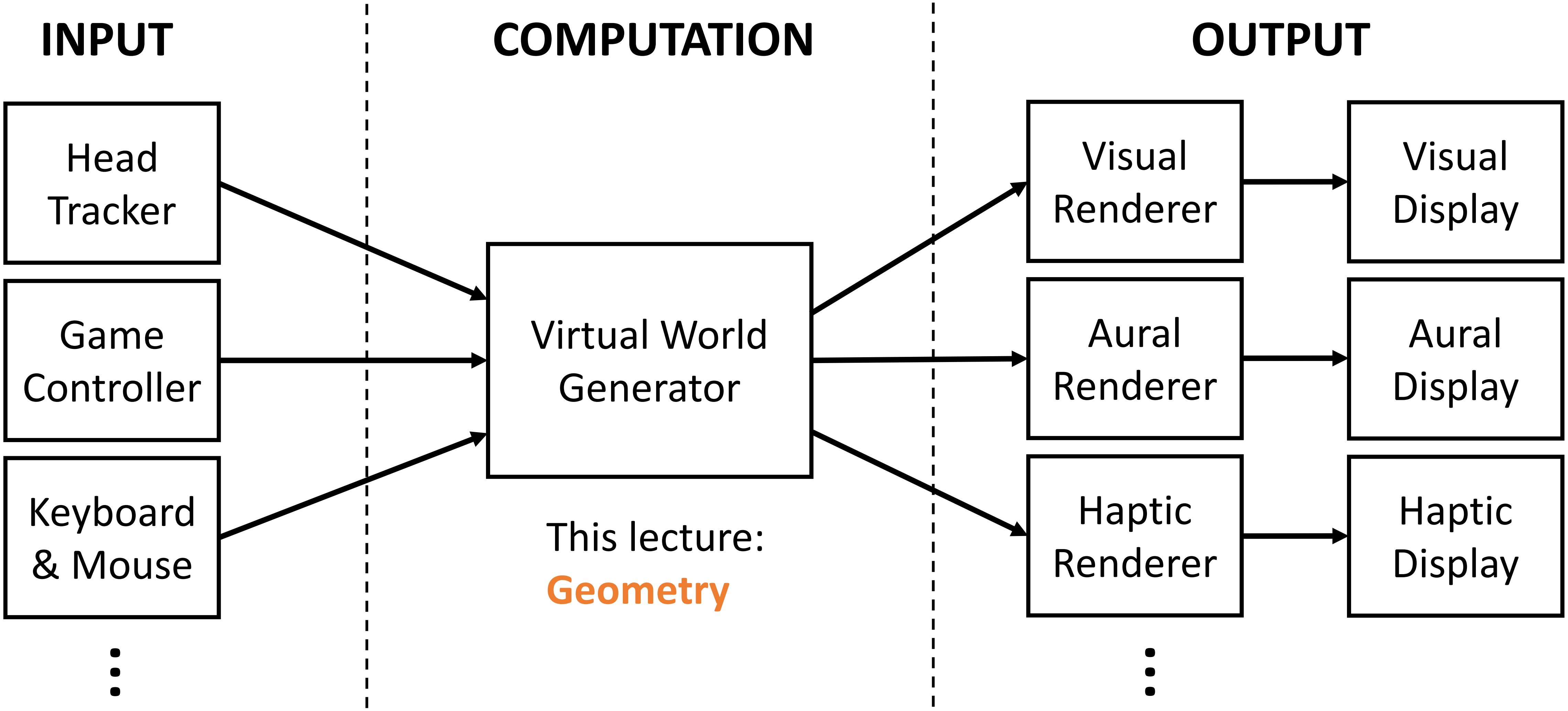
Some slides of this lecture are based on the Virtual Reality textbook by Steven LaValle

Review of VR Systems



This lecture: **Geometry**

How to Build the Virtual World?

- Computer games

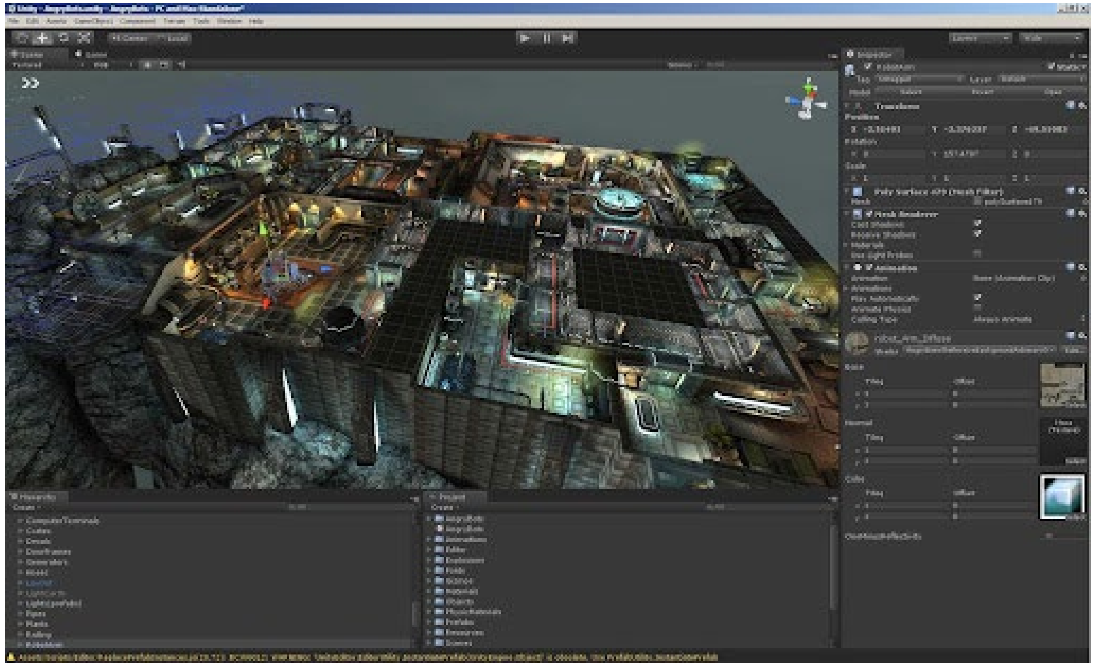
2D Virtual World

3D Virtual World

3D Virtual World, first person

How to Build the Virtual World?

- Examples of game engines

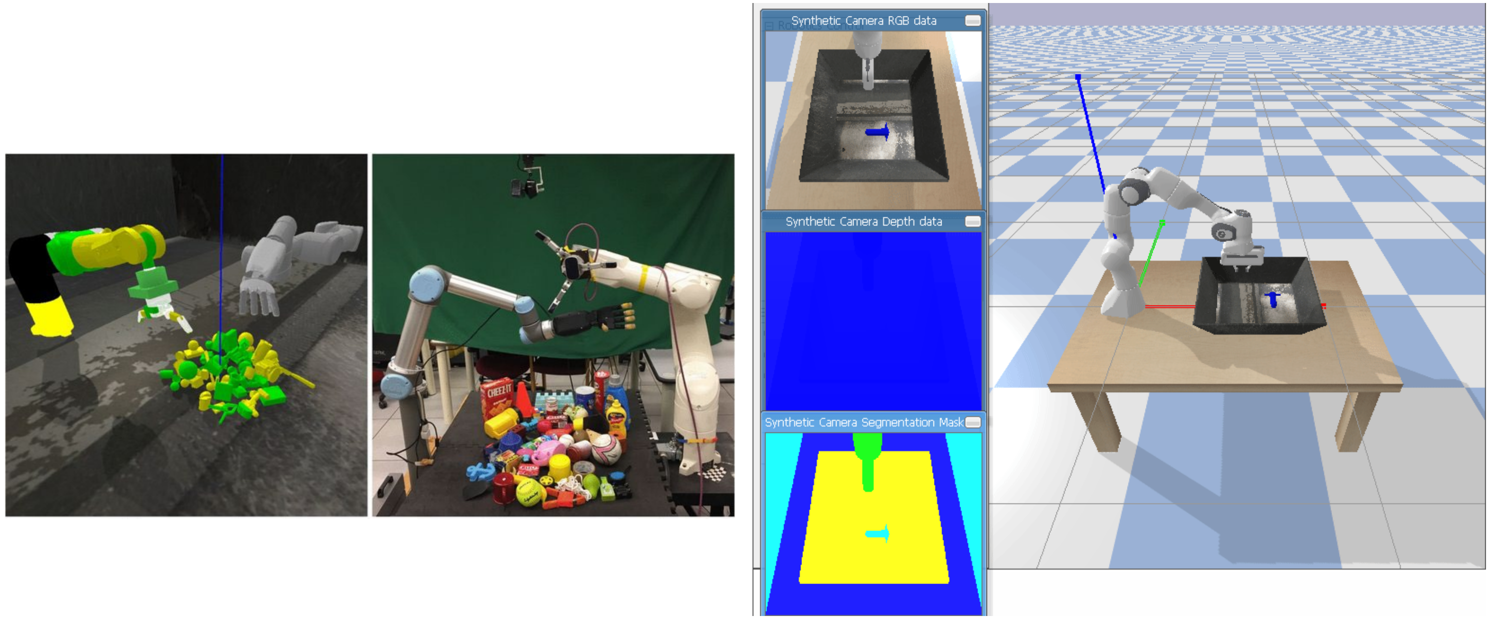


Unity

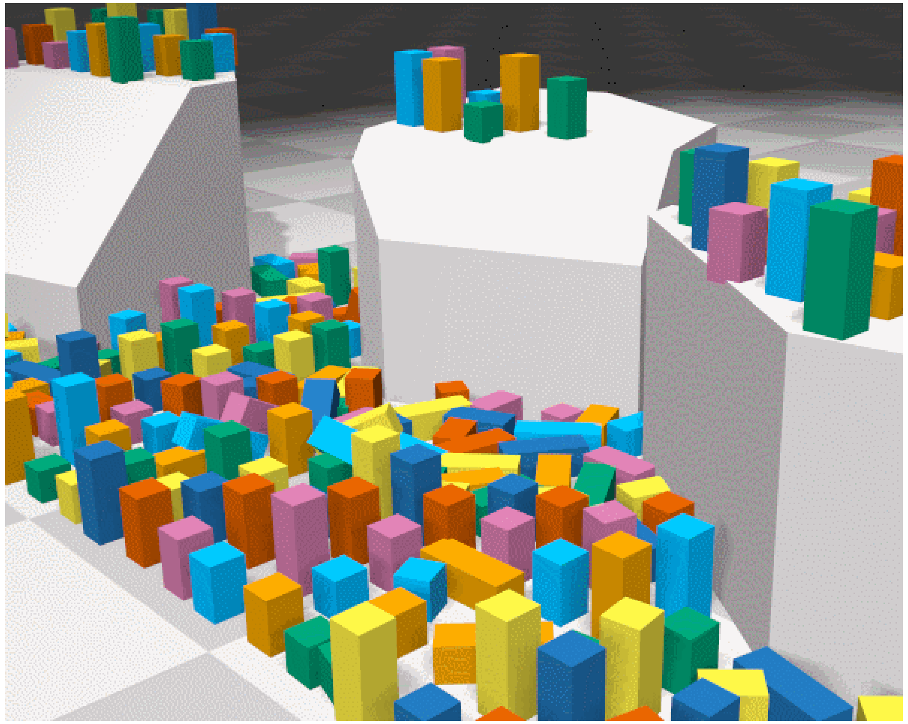
Unreal

How to Build the 3D World?

- Physics simulation



PyBullet Simulation



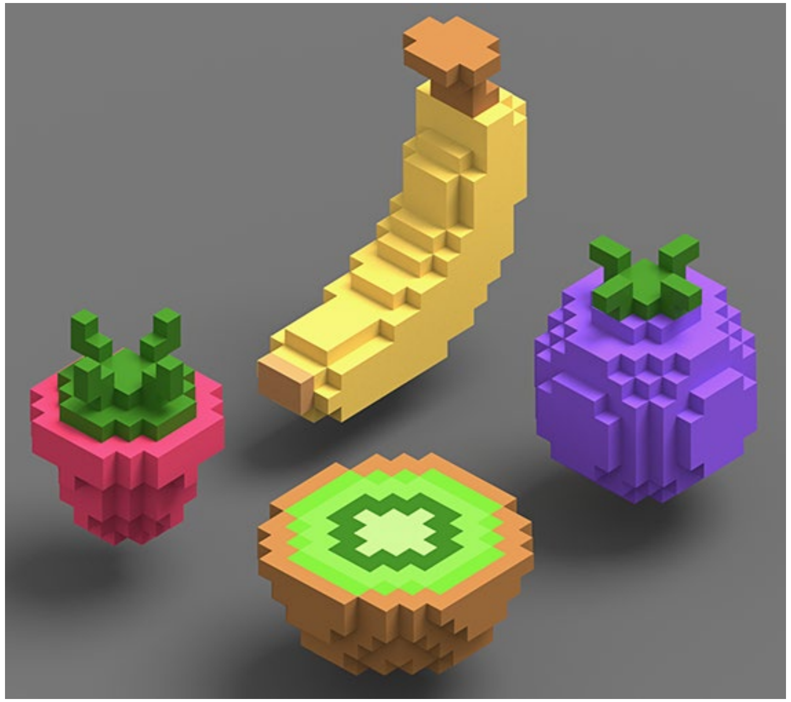
NVIDIA FleX Simulation

How to Build the 3D World?

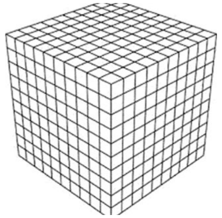
- Game engines
  - Photo-realistic rendering
  - Built in physics simulation, e.g., Unity uses the NVIDIA PhysX engine
  - Need more experience

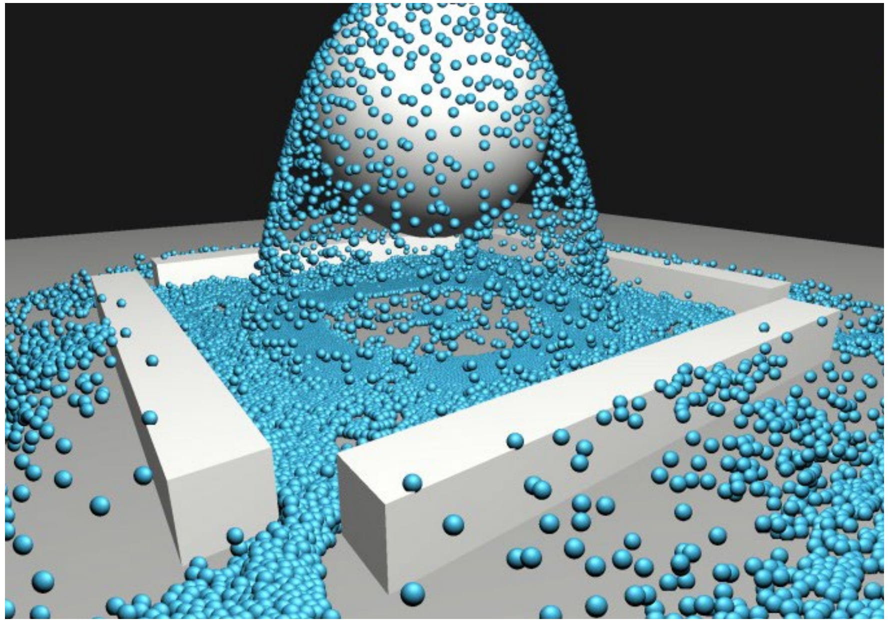
- Physics simulators
  - Usually non-photo-realistic rendering
  - Usually easy to program, e.g., PyBullet
  - Good for learning the concepts in VR

Representations of the 3D World

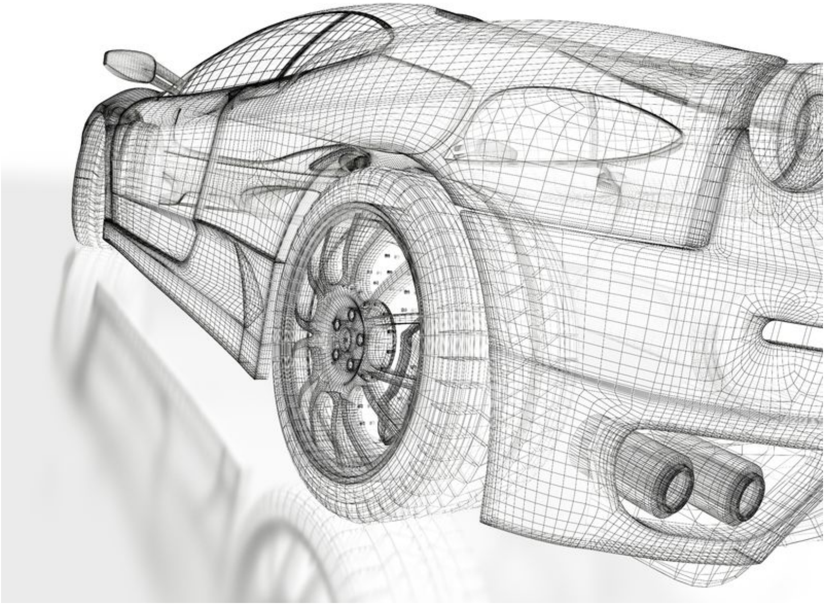


3D Voxels



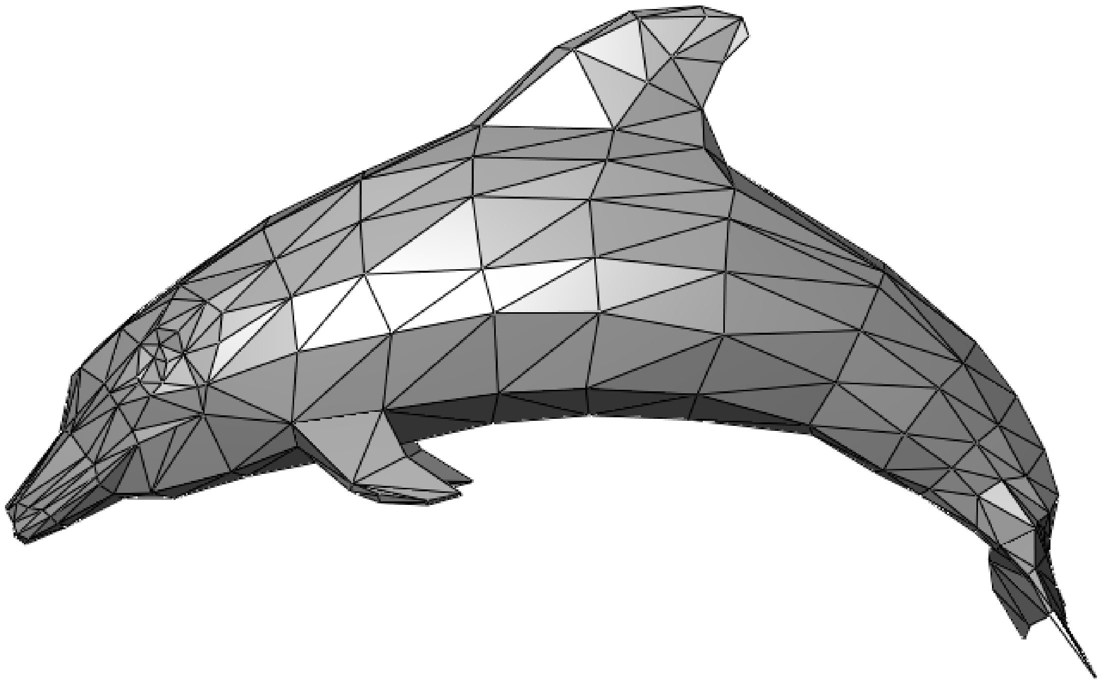


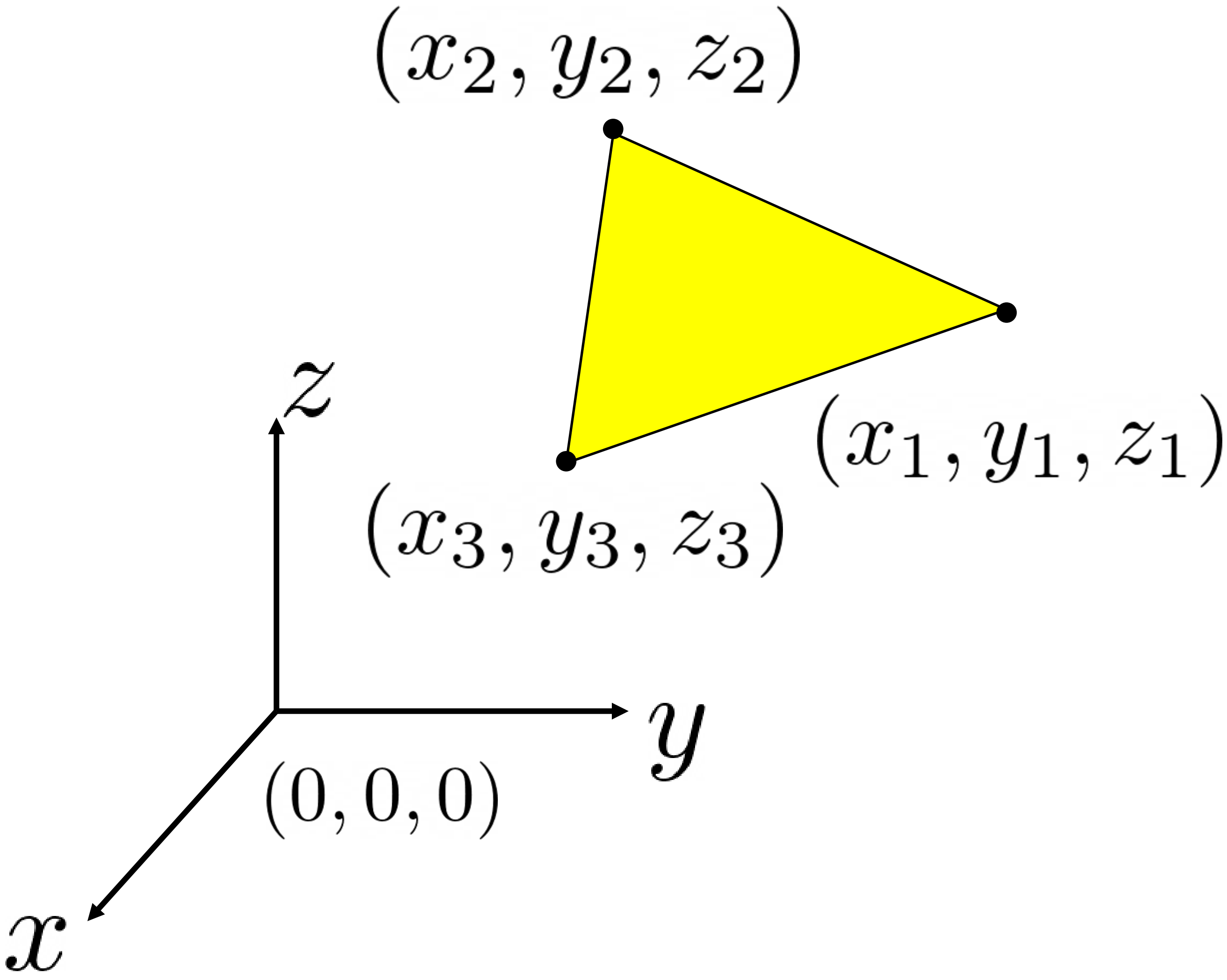
3D Particles



3D Meshes

3D Triangle Mesh





The Virtual World as 3D Triangle Meshes

Face-Vertex Meshes

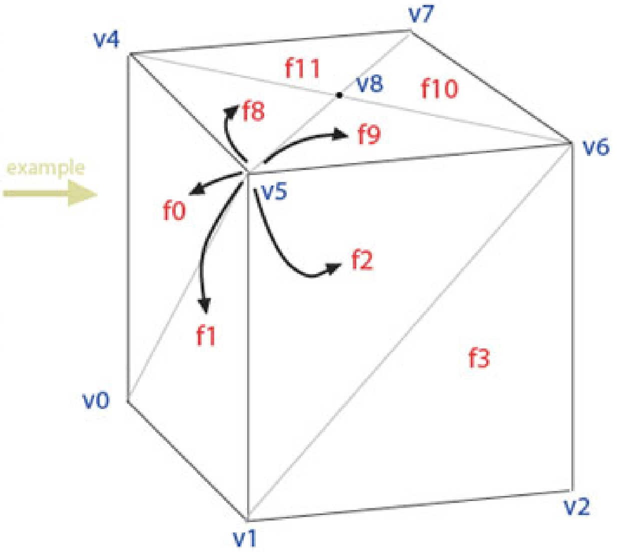
Face List

f0	v0 v4 v5
f1	v0 v5 v1
f2	v1 v5 v6
f3	v1 v6 v2
f4	v2 v6 v7
f5	v2 v7 v3
f6	v3 v7 v4
f7	v3 v4 v0
f8	v8 v5 v4
f9	v8 v6 v5
f10	v8 v7 v6
f11	v8 v4 v7
f12	v9 v5 v4
f13	v9 v6 v5
f14	v9 v7 v6
f15	v9 v4 v7

Vertex List

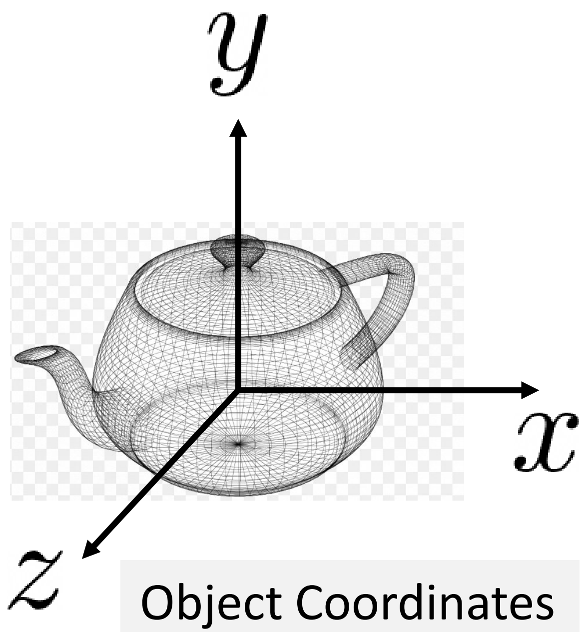
v0	0,0,0	f0 f1 f12 f15 f7
v1	1,0,0	f2 f3 f13 f12 f1
v2	1,1,0	f4 f5 f14 f13 f3
v3	0,1,0	f6 f7 f15 f14 f5
v4	0,0,1	f6 f7 f0 f8 f11
v5	1,0,1	f0 f1 f2 f9 f8
v6	1,1,1	f2 f3 f4 f10 f9
v7	0,1,1	f4 f5 f6 f11 f10
v8	.5,.5,0	f8 f9 f10 f11
v9	.5,.5,1	f12 13 14 15

example

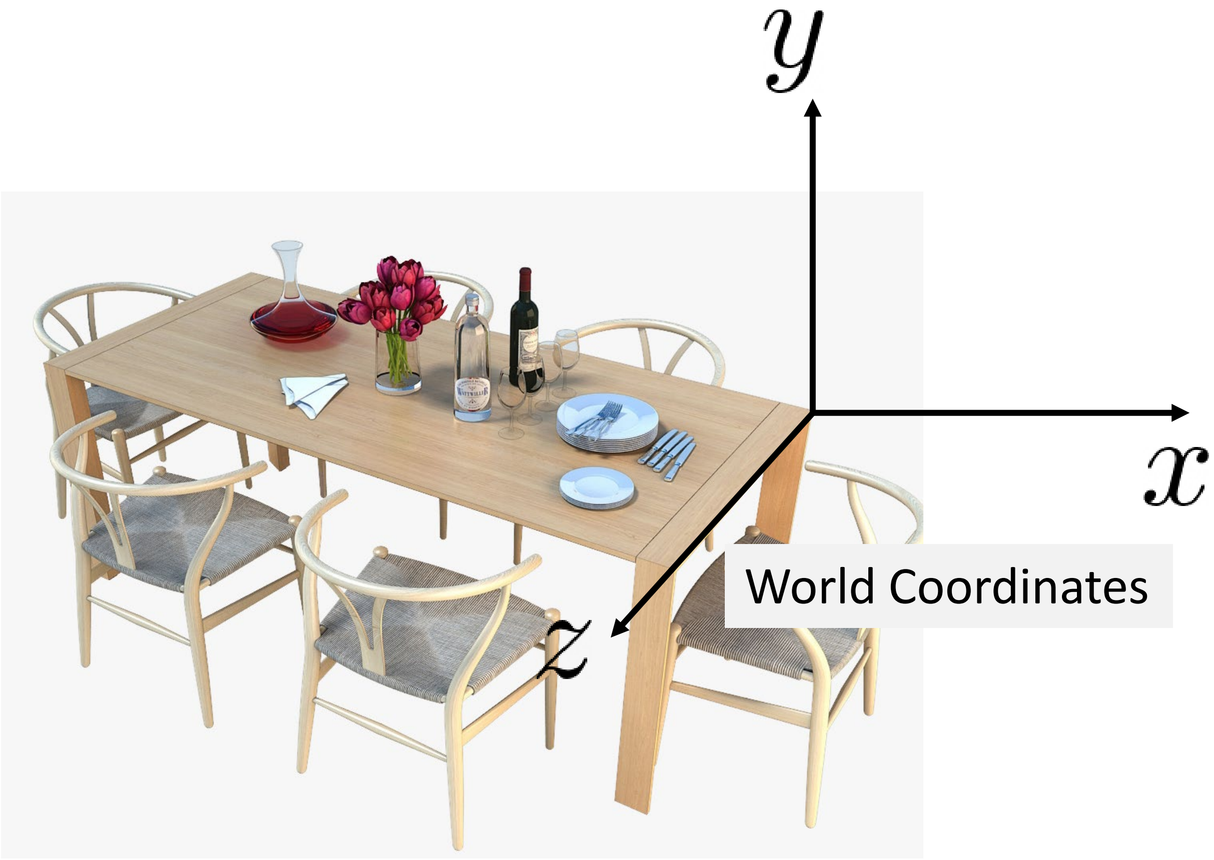


From Wikipedia

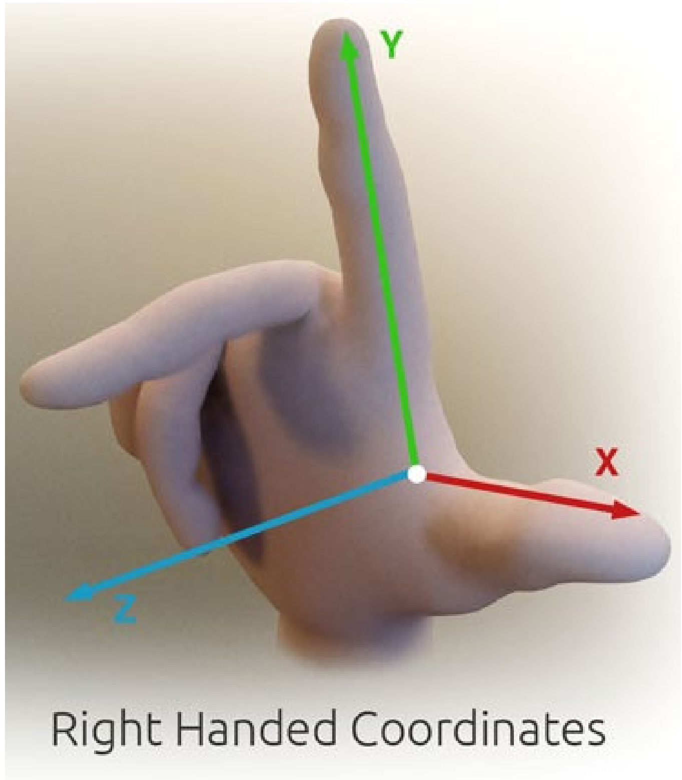
Coordinate Systems



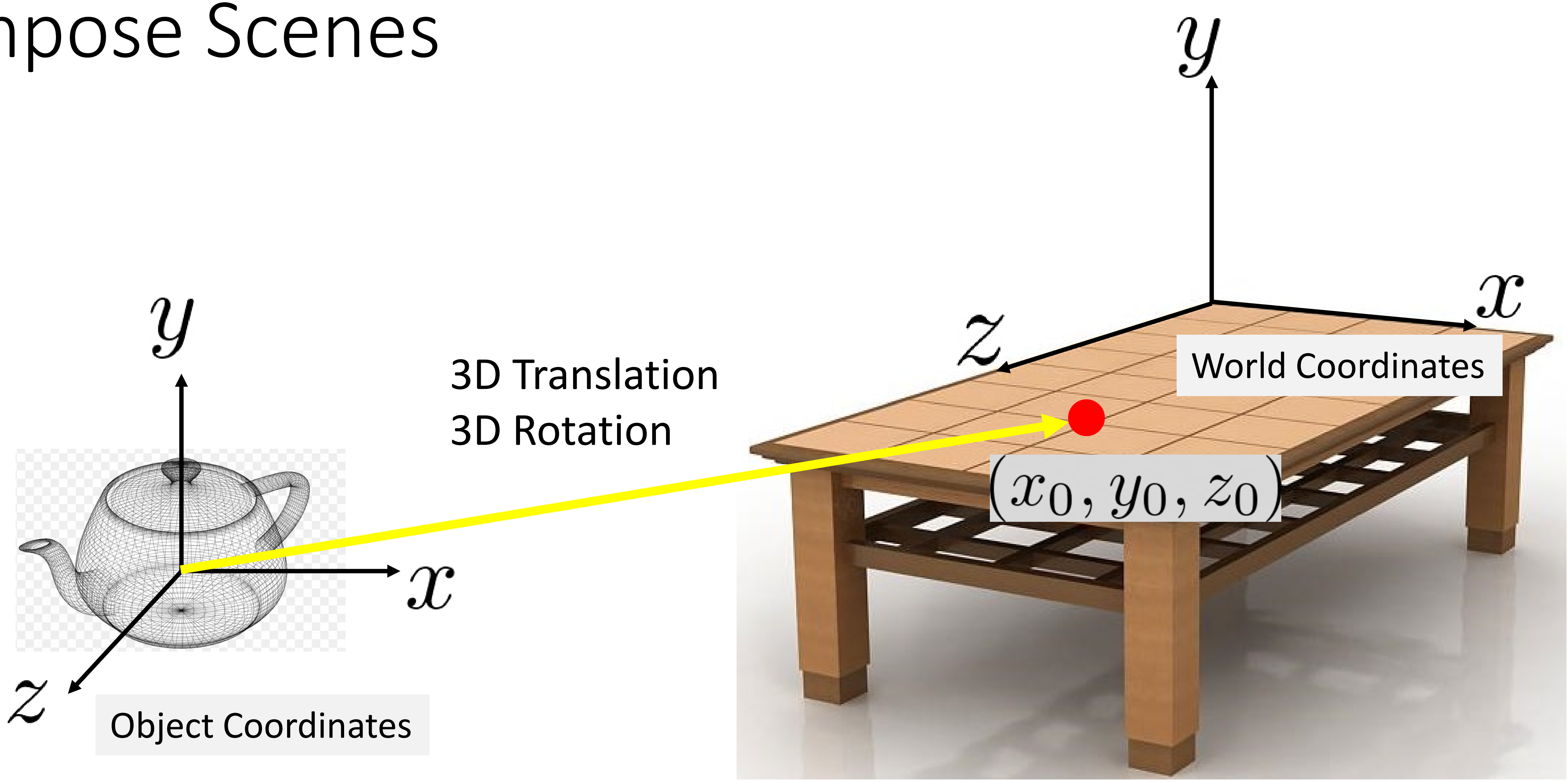
Object Coordinates



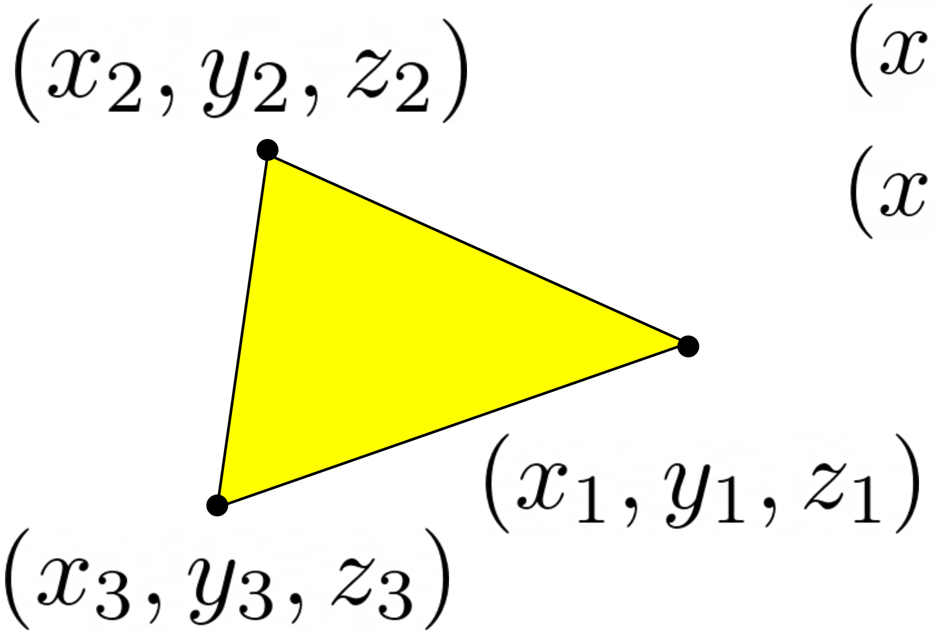
World Coordinates



Compose Scenes



3D Translation

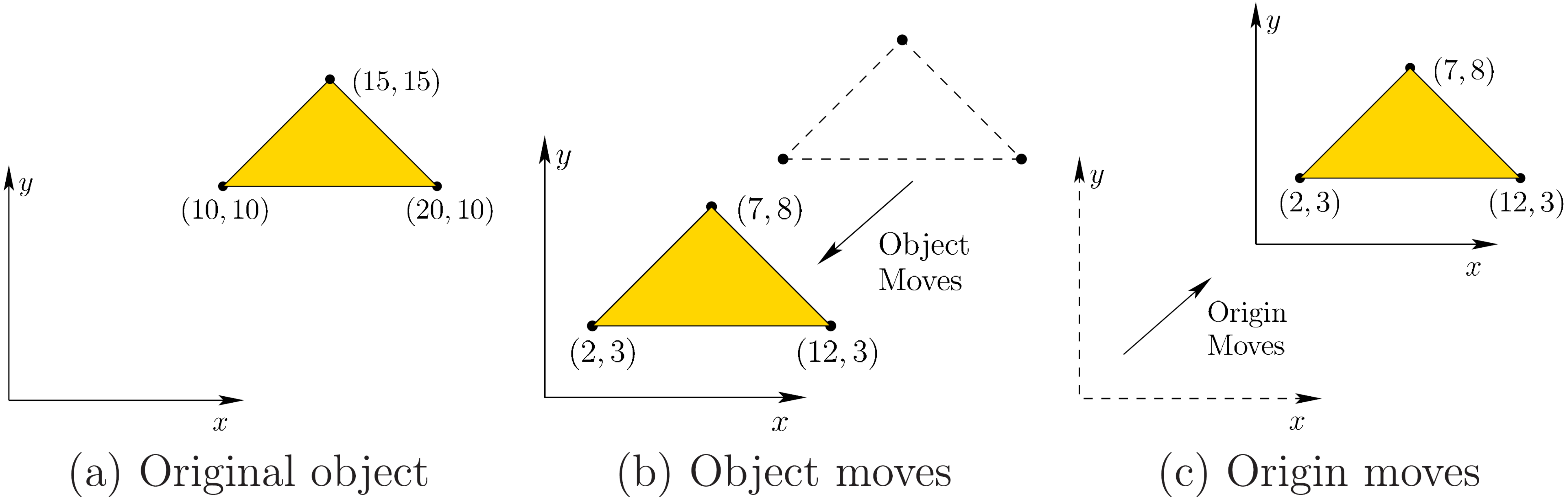


$$(x_1, y_1, z_1) \mapsto (x_1 + x_t, y_1 + y_t, z_1 + z_t)$$
$$(x_2, y_2, z_2) \mapsto (x_2 + x_t, y_2 + y_t, z_2 + z_t)$$
$$(x_3, y_3, z_3) \mapsto (x_3 + x_t, y_3 + y_t, z_3 + z_t)$$

$$\mathbf{v_1} \mapsto \mathbf{v_1} + \mathbf{t}$$
$$\mathbf{v_2} \mapsto \mathbf{v_2} + \mathbf{t}$$
$$\mathbf{v_3} \mapsto \mathbf{v_3} + \mathbf{t}$$

3D Translation $\mathbf{t} = (x_t, y_t, z_t)$

Relativity



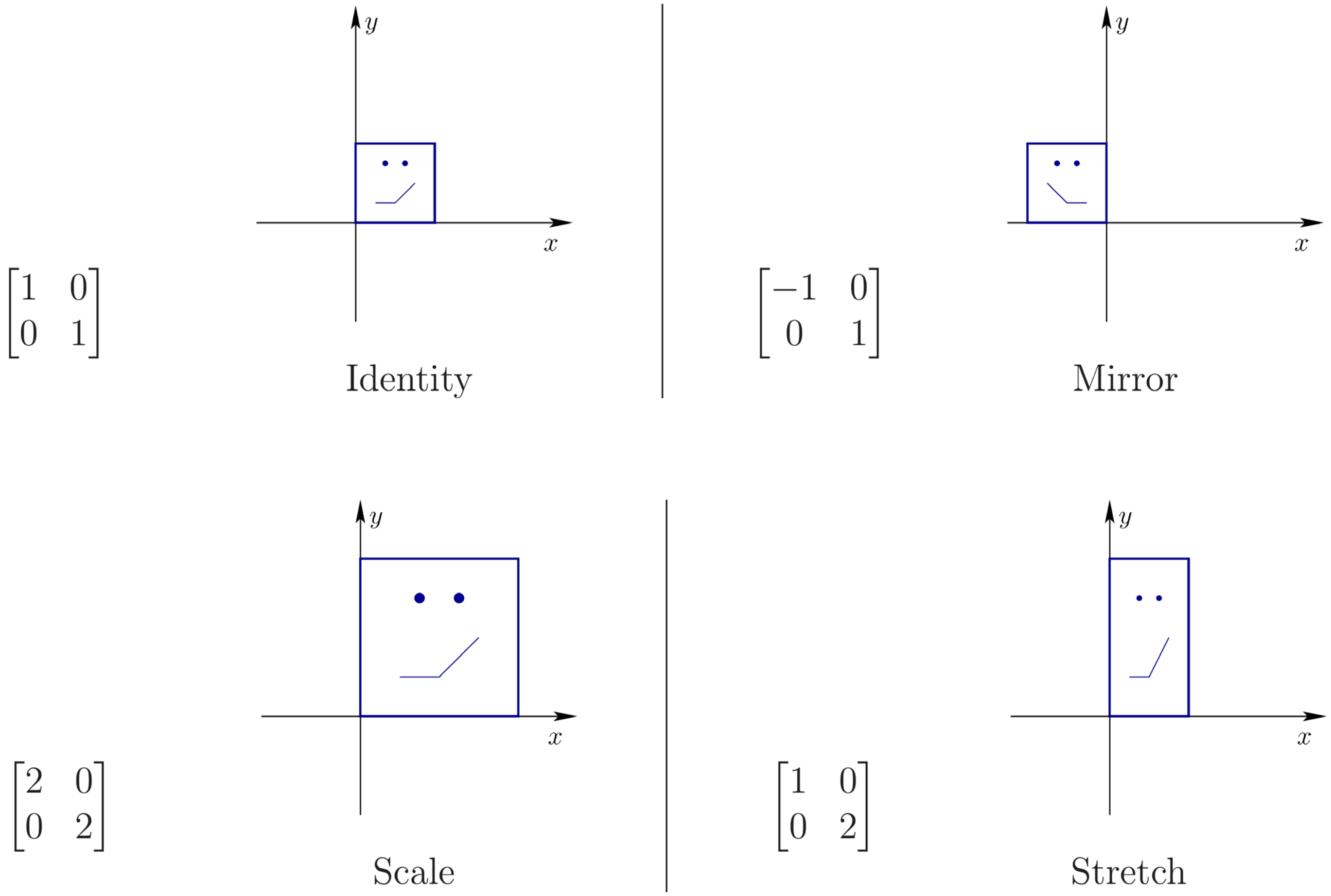
Both result in the same coordinates of the triangle

Apply a 2D Matrix to a 2D point

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x' = m_{11}x + m_{12}y$$
$$y' = m_{21}x + m_{22}y$$



Apply a 2D Matrix to a 2D point

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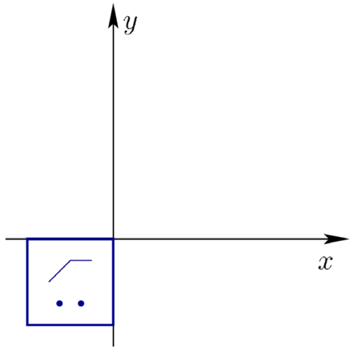
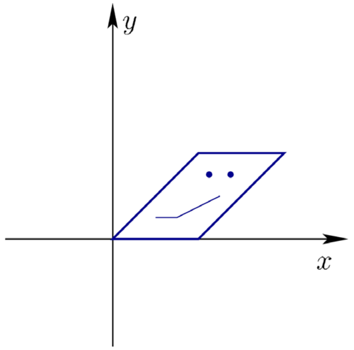
$$x' = m_{11}x + m_{12}y$$
$$y' = m_{21}x + m_{22}y$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



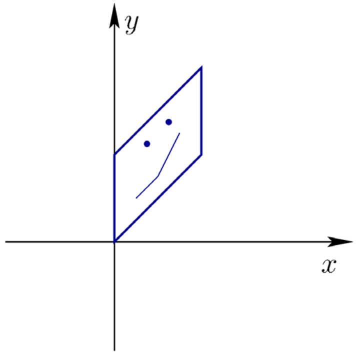
Rotate 180

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



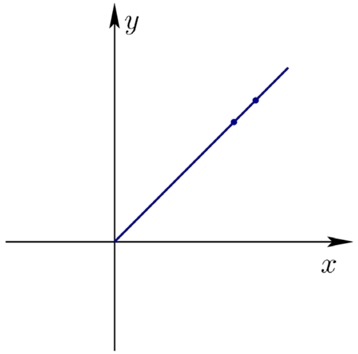
x-shear

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



y-shear

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$



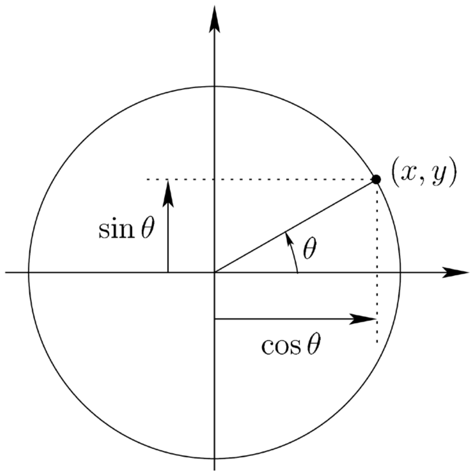
Singular

2D Rotations

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

- No stretching of axes $m_{11}^2 + m_{21}^2 = 1 \text{ and } m_{12}^2 + m_{22}^2 = 1$
- No shearing Dot product $m_{11}m_{12} + m_{21}m_{22} = 0$
- No mirror images $\det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = m_{11}m_{22} - m_{12}m_{21} = 1$

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

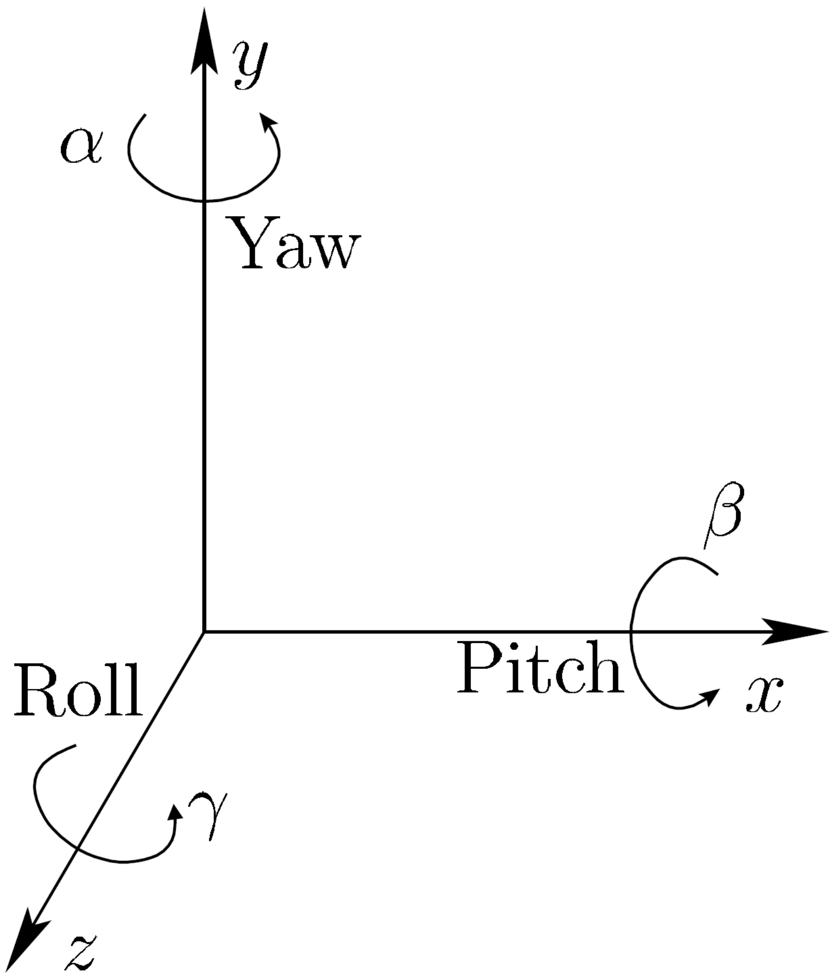


1 Degree of Freedom
Rotate by θ

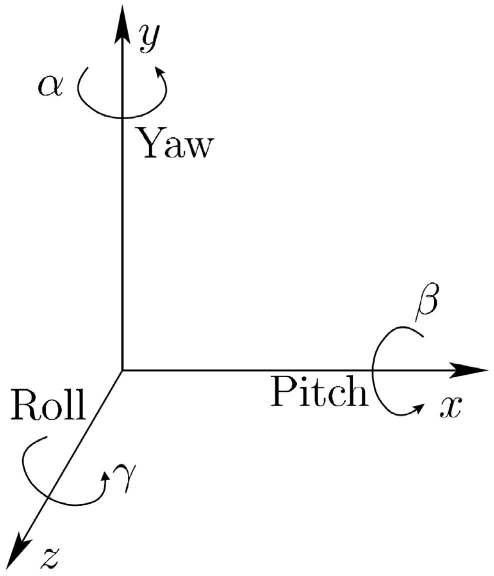
3D Rotations

- Unit-length columns
- Perpendicular columns
- $\det M = 1$
- 3 DOFs

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$



Euler Angles: Yaw, Pitch, Roll



- Counterclockwise rotation

Roll

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pitch

$$R_x(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

Yaw

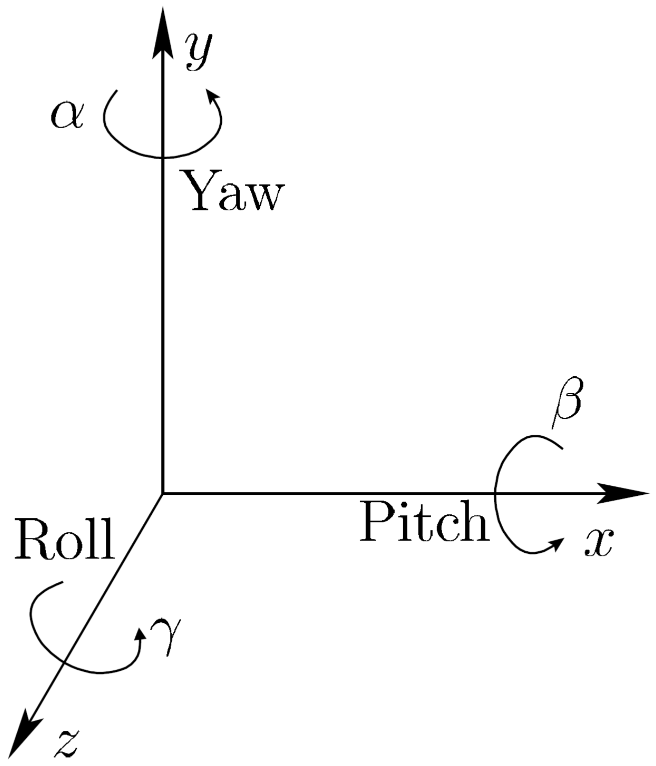
$$R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Combining Rotations

- Matrix multiplications are "backwards"

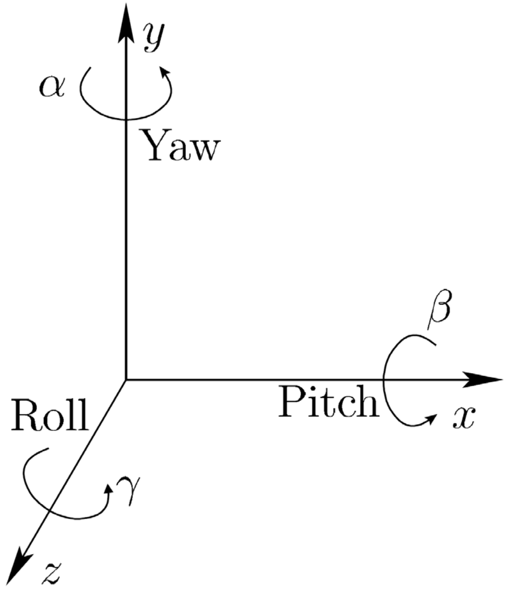
$$R(\alpha, \beta, \gamma) = R_y(\alpha) R_x(\beta) R_z(\gamma)$$

$$\alpha, \gamma \in [0, 2\pi] \qquad \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$



Singularities

- When pitch $\beta = \frac{\pi}{2}$



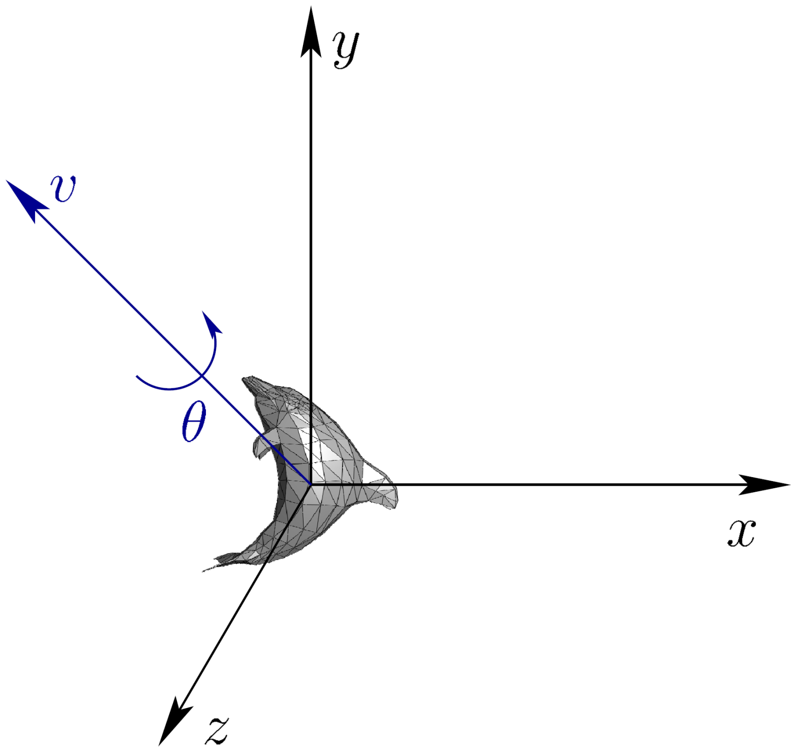
$$R(\alpha, \beta, \gamma) = R_y(\alpha) R_x(\beta) R_z(\gamma)$$

$$\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha-\gamma) & \sin(\alpha-\gamma) & 0 \\ 0 & 0 & -1 \\ -\sin(\alpha-\gamma) & \cos(\alpha-\gamma) & 0 \end{bmatrix}$$

Only one DOF

Axis-Angle Representations of Rotation

- Euler's rotation theorem: every 3D rotation can be considered as a rotation by an angle about an axis through the origin

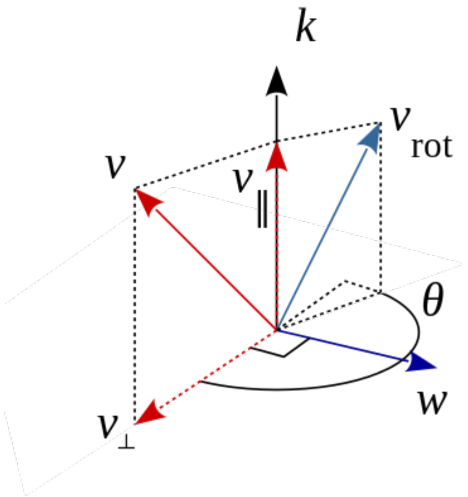


$$\mathbf{v} = (v_1, v_2, v_3)$$

Unit vector

2DOF + 1DOF

Rodrigues' Rotation Formula



- Rotate $\mathbf{v} \in \mathbb{R}^3$ about unit vector $\mathbf{k}$ by angle θ

$$\mathbf{v}_{\text{rot}} = \mathbf{v}\cos\theta + (\mathbf{k} \times \mathbf{v})\sin\theta + \mathbf{k}(\mathbf{k} \cdot \mathbf{v})(1 - \cos\theta)$$

Derivation HW1

- Matrix notation $\mathbf{v}_{\text{rot}} = \mathbf{R}\mathbf{v}$

$$\mathbf{R} = \mathbf{I} + (\sin\theta)\mathbf{K} + (1 - \cos\theta)\mathbf{K}^2$$

Cross product matrix

$$\mathbf{K} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

$$\mathbf{k} \times \mathbf{v} = \mathbf{K}\mathbf{v}$$

https://en.wikipedia.org/wiki/Cross_product

SO(n): Special Orthogonal Group

- SO(n): Space of rotation matrices in $\mathbb{R}^n$

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : RR^T = I, \det(R) = 1\}$$

- SO(3): space of 3D rotation matrices
- Group is a set G, with an operation ●, satisfying the following axioms:
  - Closure: $a \in G, b \in G \Rightarrow a \cdot b \in G$
  - Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c), \forall a, b, c \in G$
  - Identity element: $\exists e \in G, e \cdot a = a, \forall a \in G$
  - Inverse element: $\forall a \in G, \exists b \in G, a \cdot b = b \cdot a = e$

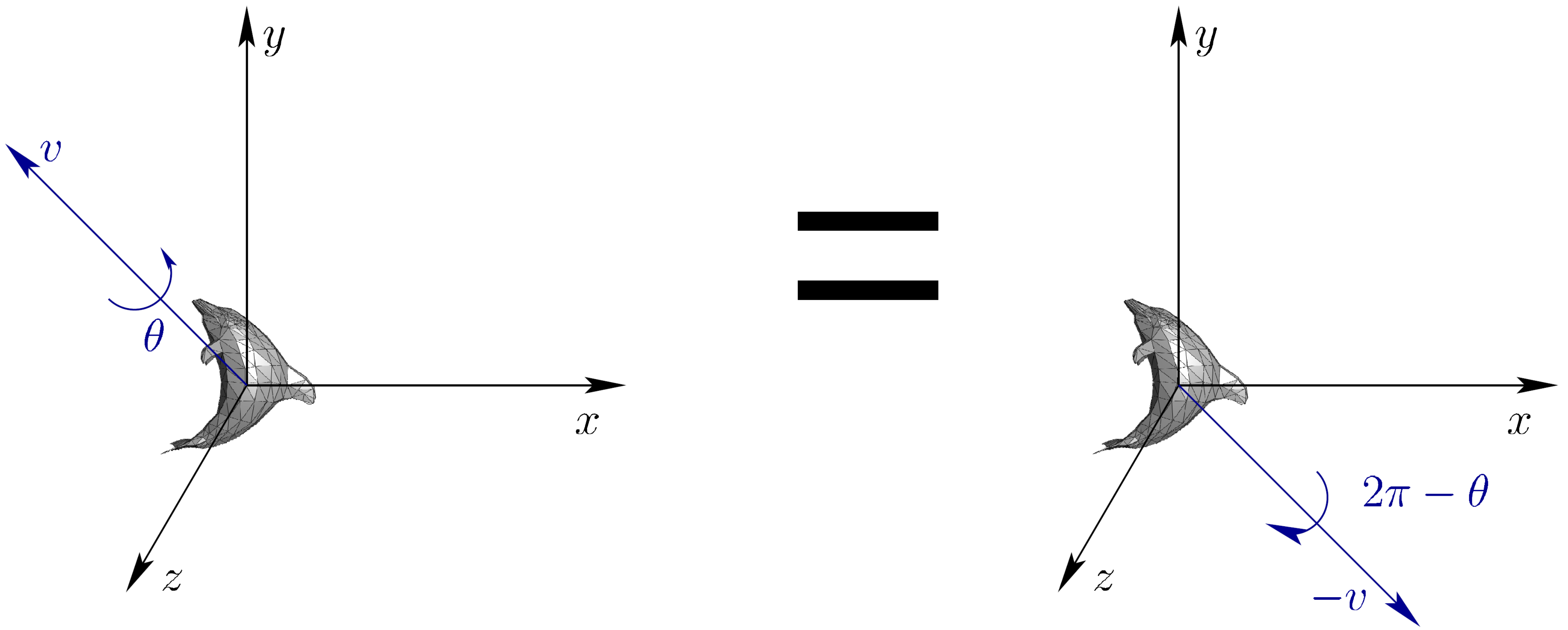
Exponential Map for SO(3)

- Matrix exponential $\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$ factorial
- For Lie Group, Hamilton-Cayley theorem $\exp(X) = \sum_{k=0}^{n-1} a_k(X) X^k$
  - Coefficients are functions of eigenvalues of X $\quad X : n \times n$
- For SO(3), Rodrigues' Rotation Formula $so(n) = \{K \in \mathbb{R}^{n \times n} : K^T = -K\}$

skew-symmetric matrix $\mathbf{K} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$

$$\mathbf{R} = \mathbf{I} + (\sin\theta)\mathbf{K} + (1 - \cos\theta)\mathbf{K}^2$$
$$= \exp(\theta K)$$

Two-to-one Problem of Axis-Angle Representations



Quaternions for 3D Rotations

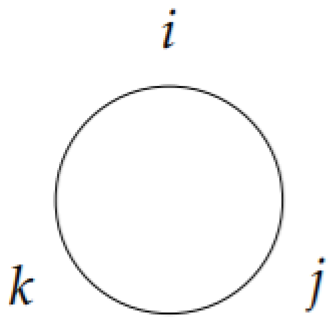
- Quaternions generalize complex numbers and can be used to represents 3D rotations

$$q = w + \underbrace{xi + yj + zk}$$

Scale (real part) Vector (imaginary part)

- Properties

$$i^2 = j^2 = k^2 = -1$$
$$ij = k, ji = -k$$
$$jk = i, kj = -i$$
$$ki = j, ik = -j$$



Quaternion Addition and Multiplication

- Addition

$$p + q = (p_0 + q_0) + (p_1 + q_1)\boldsymbol{i} + (p_2 + q_2)\boldsymbol{j} + (p_3 + q_3)\boldsymbol{k}$$

- Multiplication

$$\begin{aligned} pq &= (p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k})(q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \\ &= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + p_0(q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) + q_0(p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}) \\ &\qquad + (p_2q_3 - p_3q_2)\boldsymbol{i} + (p_3q_1 - p_1q_3)\boldsymbol{j} + (p_1q_2 - p_2q_1)\boldsymbol{k}. \end{aligned}$$

$$pq = p_0q_0 - \boldsymbol{p} \cdot \boldsymbol{q} + p_0\boldsymbol{q} + q_0\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{q}$$

$$\boldsymbol{p} = (p_1, p_2, p_3) \quad \boldsymbol{q} = (q_1, q_2, q_3)$$

Complex Conjugate, Norm and Inverse

- Conjugate $$q = q_0 + \boldsymbol{q} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$$
$$q^* = q_0 - \boldsymbol{q} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}$$

- Norm $$|q| = \sqrt{q^* q}$$
$$\begin{aligned} q^* q &= (q_0 - \boldsymbol{q})(q_0 + \boldsymbol{q}) \\ &= q_0 q_0 - (-\boldsymbol{q}) \cdot \boldsymbol{q} + q_0 \boldsymbol{q} + (-\boldsymbol{q}) q_0 + (-\boldsymbol{q}) \times \boldsymbol{q} \\ &= q_0^2 + \boldsymbol{q} \cdot \boldsymbol{q} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \\ &= q q^*. \end{aligned}$$

- Inverse $$q^{-1} = \frac{q^*}{|q|^2} \qquad q^{-1} q = q q^{-1} = 1$$

Unit Quaternions as 3D Rotations

- For $\boldsymbol{v} \in \mathbb{R}^3$, rotation according to a unit quaternion $q = q_0 + \boldsymbol{q}$

$$\begin{aligned} L_q(\boldsymbol{v}) &= q\boldsymbol{v}q^* \\ &= (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q} \cdot \boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \end{aligned}$$

The real part of v is 0

- For unit quaternions, axis-angle

$$(v, \theta) \longleftrightarrow q = \left(\cos\frac{\theta}{2},\ v_1 \sin\frac{\theta}{2},\ v_2 \sin\frac{\theta}{2},\ v_3 \sin\frac{\theta}{2}\right)$$

Two Equivalent Quaternions for 3D Rotation

- Multiply -1 to a quaternion

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\frac{\vec{u}}{||\vec{u}||}$$

$$-q = \cos\left(\frac{\theta}{2} + \pi\right) + \sin\left(\frac{\theta}{2} + \pi\right)\frac{\vec{u}}{||\vec{u}||}$$

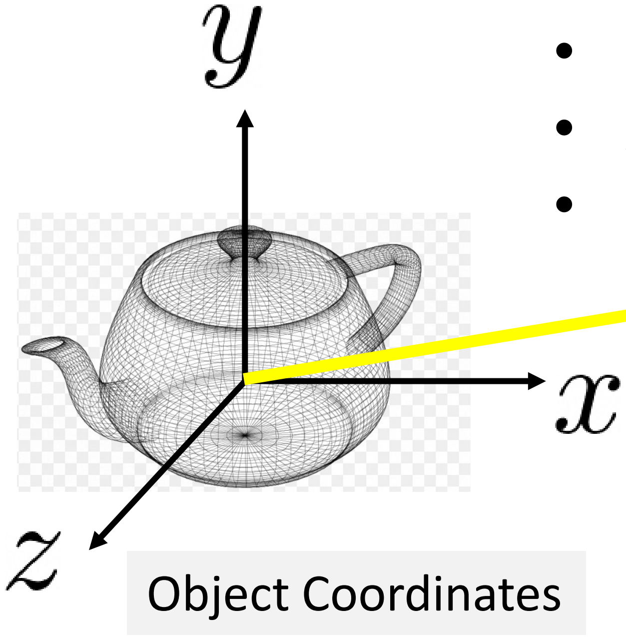
- q rotates θ , $-q$ rotates $\theta + 2\pi$

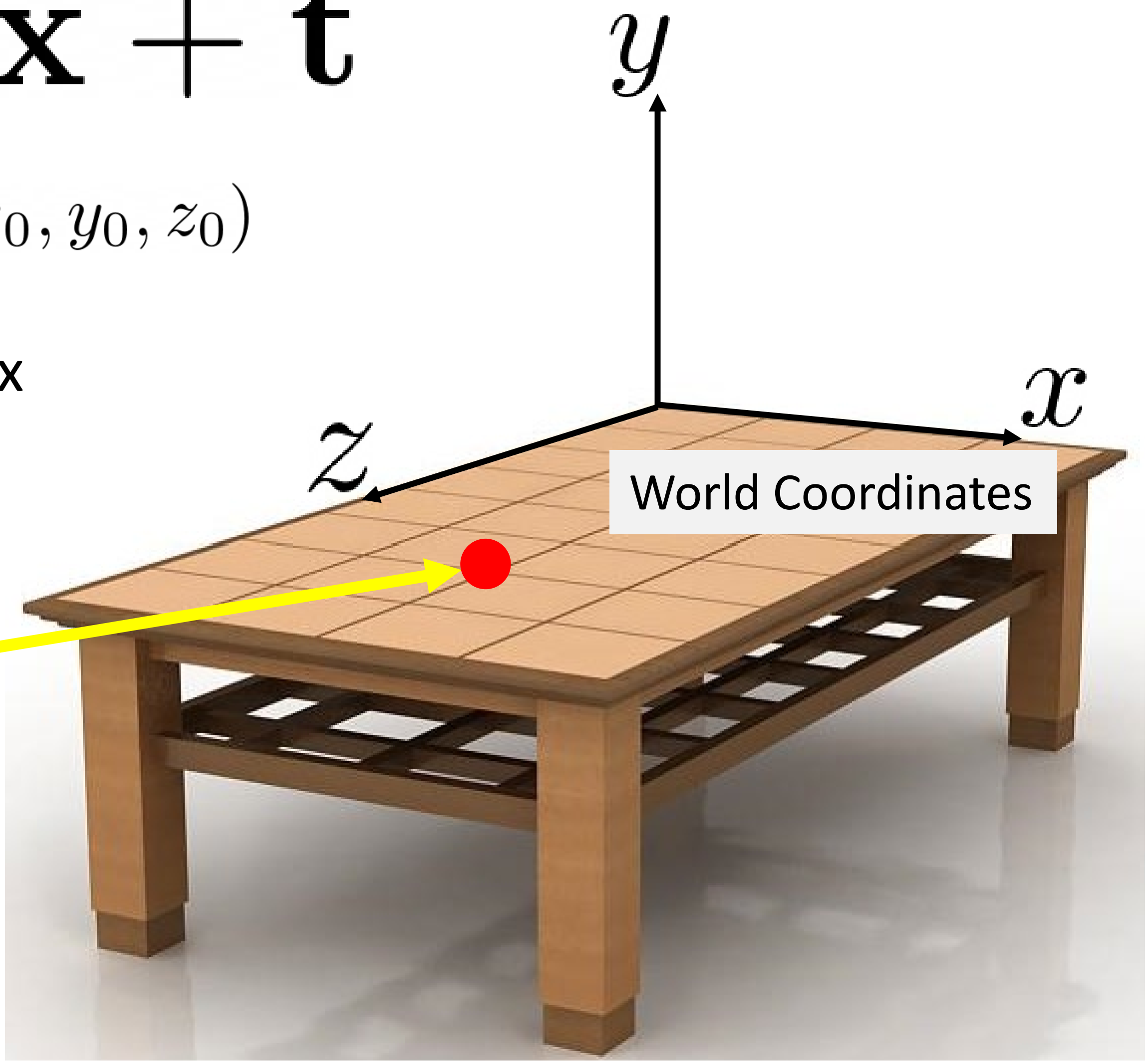
Transform 3D Models $R\mathbf{x} + \mathbf{t}$

3D Translation (x_0, y_0, z_0)

3D Rotation

- Rotation matrix
- Euler angles
- Axis-angle
- Quaternion





Further Reading

- Chapter 3, Virtual Reality, Steven LaValle
- Quaternion and Rotations, Yan-Bin Jia, https://graphics.stanford.edu/courses/cs348a-17-winter/Papers/quaternion.pdf
- Introduction to Robotics, Prof. Wei Zhang, OSU, Lecture 3, Rotational Motion, http://www2.ece.ohio-state.edu/~zhang/RoboticsClass/index.html