Dynamics of Open Chains

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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Robot Dynamics

- Study motion of robots with the forces and torques that cause them
  - Newton's second law F = ma

- Forward dynamics
  - Given robot state $(\theta, \dot{\theta})$ and the joint forces and torques τ
  - Determine the robot's acceleration $\ddot{\theta}$

Simulation

- Inverse dynamics
  - Given robot state $(\theta, \dot{\theta})$ and a desired acceleration $\ddot{\theta}$ (from motion planning)
  - Find the joint forces and torques τ

Control

Dynamics of a Single Rigid Body

- Inverse dynamics $$\mathcal{F}_b \;=\; \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b$$
- Forward dynamics $$\dot{\mathcal{V}}_b = \mathcal{G}_b^{-1}(\mathcal{F}_b + [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b)$$

Body wrench $\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix}$

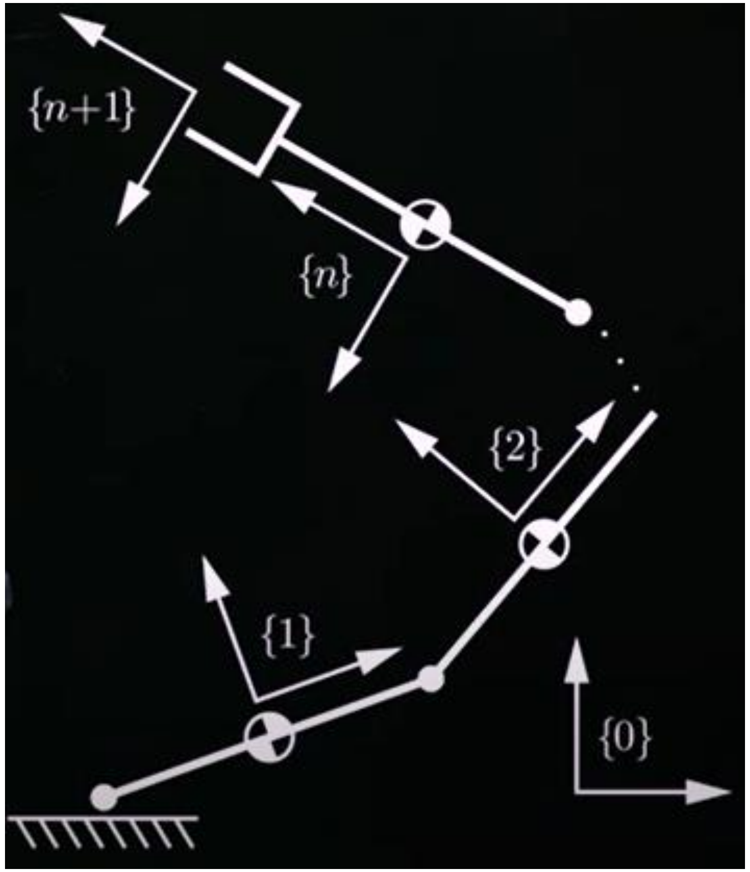
Body twist $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$

Spatial inertia matrix $\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m} I \end{bmatrix}$

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Inverse Dynamics of Open Chains

- N-link open chain
- A body-fixed reference frame {i} is attached to the center of mass of each link i
- Base frame {0}, end-effector frame {n+1} (fixed in {n})

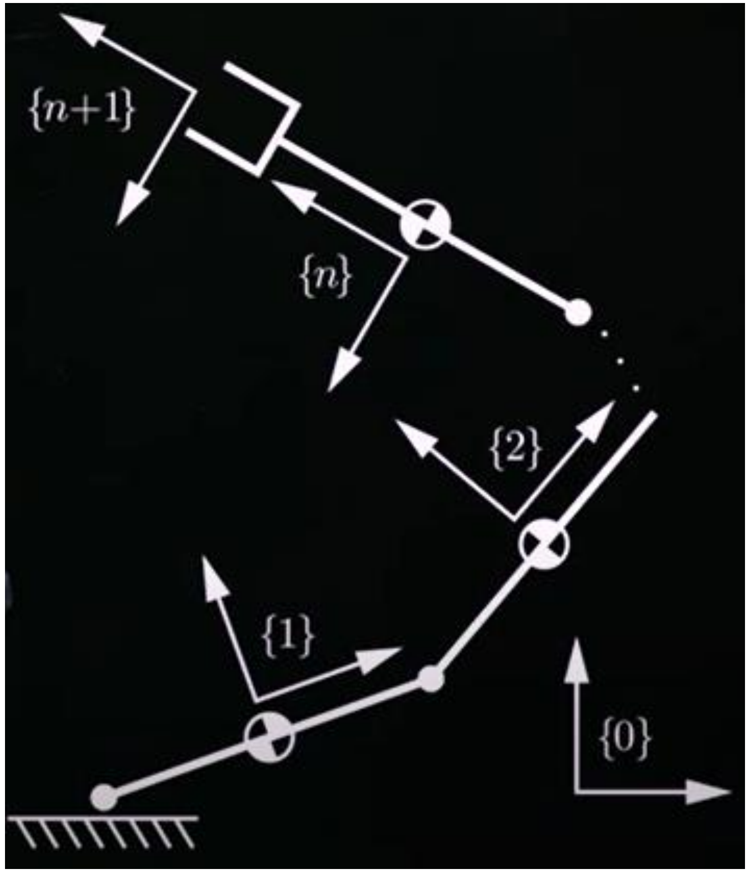


Inverse Dynamics of Open Chains

- At home position (all joints are zeros)
  - Configuration of frame {j} in {i} $M_{i,j} \in SE(3)$
  - Configuration of {i} in base frame {0} $M_i = M_{0,i}$

$$M_{i-1,i} = M_{i-1}^{-1} M_i$$

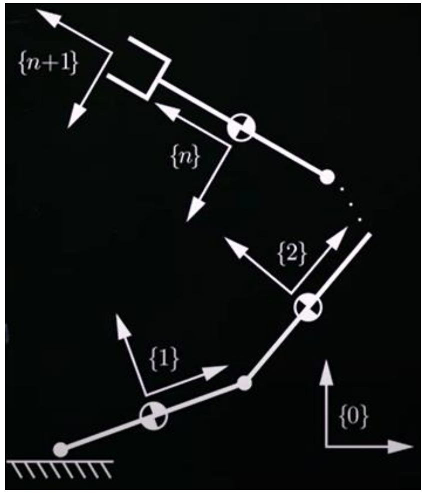
$$M_{i,i-1} = M_i^{-1} M_{i-1}$$



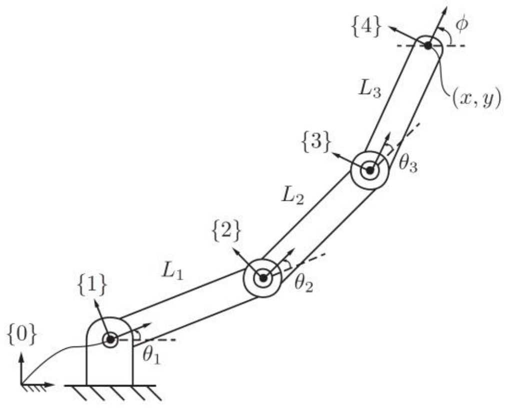
Inverse Dynamics of Open Chains

- Screw axis for joint i in link frame {i} $\mathcal{A}_i$, in space frame {0} $\mathcal{S}_i$

$$\mathcal{A}_i = \mathrm{Ad}_{M_i^{-1}}(\mathcal{S}_i)$$



- Recall screw axis

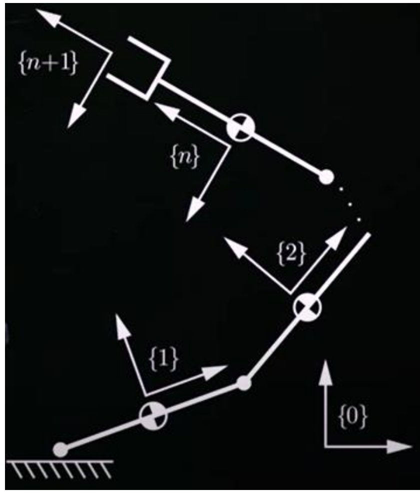


$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

$$\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b \qquad [\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Inverse Dynamics of Open Chains

- Screw axis for joint i in link frame {i} $\mathcal{A}_i$, in space frame {0} $\mathcal{S}_i$



- The configuration of {j} in {i} with joint variables $T_{i,j} \in SE(3)$

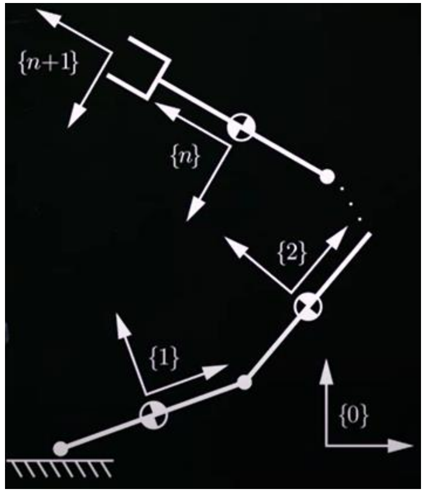
$$T_{i-1,i}(\theta_i) \qquad T_{i,i-1}(\theta_i) = T^{-1}_{i-1,i}(\theta_i)$$

$$T_{i-1,i}(\theta_i) = M_{i-1,i}e^{[\mathcal{A}_i]\theta_i} \quad T_{i,i-1}(\theta_i) = e^{-[\mathcal{A}_i]\theta_i}M_{i,i-1}$$

- Twist of link frame {i} expressed in {i} $\mathcal{V}_i = (\omega_i, v_i)$
- Wrench transmitted through joint i to link frame {i} expressed in {i}

$$\mathcal{F}_i = (m_i, f_i)$$

Inverse Dynamics of Open Chains



- Spatial inertia matrix of link i $\mathcal{G}_i \in \mathbb{R}^{6\times 6}$ $\mathcal{G}_i = \begin{bmatrix} \mathcal{I}_i & 0 \\ 0 & \mathfrak{m}_i I \end{bmatrix}$

- Recursively calculate the twist and acceleration, moving from the base to the tip

$$\mathcal{V}_i = \mathcal{A}_i \dot{\theta}_i + [\mathrm{Ad}_{T_{i,i-1}}]\mathcal{V}_{i-1}$$ (Velocity for link i)

$$\dot{\mathcal{V}}_i = \mathcal{A}_i \ddot{\theta}_i + [\mathrm{Ad}_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + \frac{d}{dt}\left([\mathrm{Ad}_{T_{i,i-1}}]\right)\mathcal{V}_{i-1}$$

See Lynch & Park for derivation

$$\dot{\mathcal{V}}_i = \mathcal{A}_i \ddot{\theta}_i + [\mathrm{Ad}_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + [\mathrm{ad}_{\mathcal{V}_i}]\mathcal{A}_i \dot{\theta}_i$$

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Inverse Dynamics of Open Chains

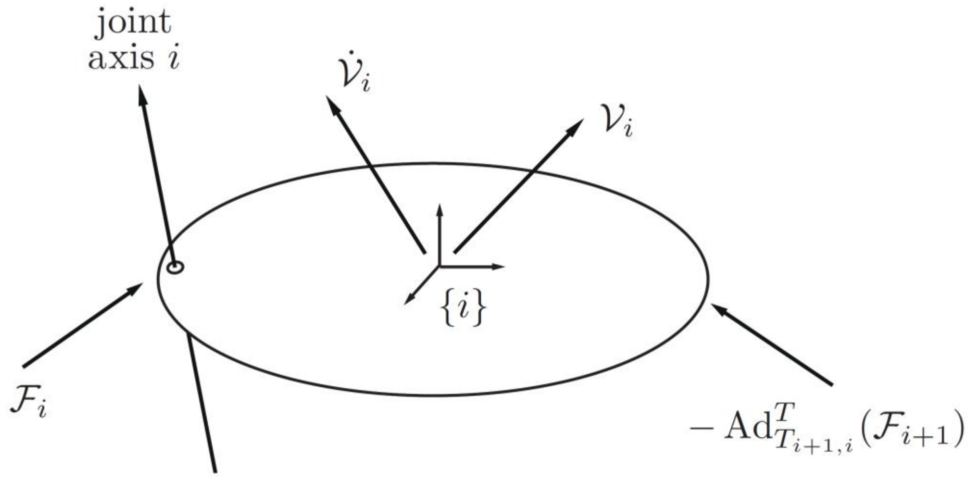
$$\mathcal{F}_b = [\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}} \mathcal{F}_a$$

- Accelerations from base to tip

$$\dot{\mathcal{V}}_i = \mathcal{A}_i \ddot{\theta}_i + [\mathrm{Ad}_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + [\mathrm{ad}_{\mathcal{V}_i}]\mathcal{A}_i \dot{\theta}_i$$

- Recall rigid body dynamic equations

$$\begin{aligned} \mathcal{F}_b &= \mathcal{G}_b \dot{\mathcal{V}}_b - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_b}(\mathcal{P}_b) \\ &= \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b \end{aligned}$$



- Wrench on link i from joint i and joint i+1

$$\mathcal{G}_i \dot{\mathcal{V}}_i - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_i}(\mathcal{G}_i \mathcal{V}_i) = \mathcal{F}_i - \mathrm{Ad}^{\mathrm{T}}_{T_{i+1,i}}(\mathcal{F}_{i+1})$$

Inverse Dynamics

- Solve the wrench from tip to base $\mathcal{F}_i$

- Force or torque at the joint in the direction of the joint's screw axis

$$\tau_i \dot{\theta}_i = \mathcal{F}_i^{\mathrm{T}} \mathcal{A}_i \dot{\theta}_i$$

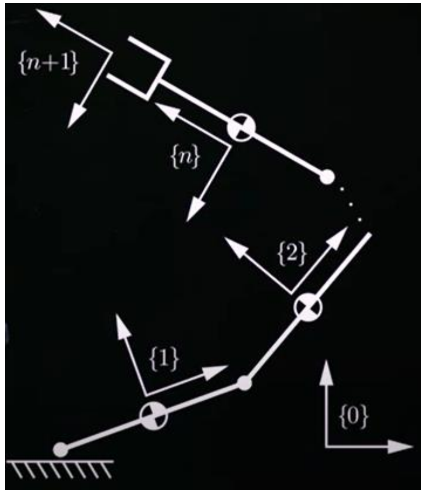
Principle of conservation of power

$$\tau_i = \mathcal{F}_i^{\mathrm{T}} \mathcal{A}_i$$

- Newton-Euler Inverse Dynamics Algorithm

Newton-Euler Inverse Dynamics Algorithm

Given $\theta, \dot{\theta}, \ddot{\theta}$ Compute τ



Forward iterations Given $\theta, \dot{\theta}, \ddot{\theta}$, for $i = 1$ to n do

$$\mathcal{V}_0 = (0,0)$$

$$\dot{\mathcal{V}}_0 = (0,-g)$$

$$\begin{aligned} T_{i,i-1} &= e^{-[\mathcal{A}_i]\theta_i} M_{i,i-1}, \\ \mathcal{V}_i &= \mathrm{Ad}_{T_{i,i-1}}(\mathcal{V}_{i-1}) + \mathcal{A}_i\dot{\theta}_i, \\ \dot{\mathcal{V}}_i &= \mathrm{Ad}_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + \mathrm{ad}_{\mathcal{V}_i}(\mathcal{A}_i)\dot{\theta}_i + \mathcal{A}_i\ddot{\theta}_i. \end{aligned}$$

Backward iterations For $i = n$ to 1 do

$$\mathcal{F}_{n+1} = \mathcal{F}_{\text{tip}} = (m_{\text{tip}}, f_{\text{tip}})$$

The wrench applied to the environment by the end-effector

$$\begin{aligned} \mathcal{F}_i &= \mathrm{Ad}^{\mathrm{T}}_{T_{i+1,i}}(\mathcal{F}_{i+1}) + \mathcal{G}_i\dot{\mathcal{V}}_i - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_i}(\mathcal{G}_i\mathcal{V}_i), \\ \tau_i &= \mathcal{F}_i^{\mathrm{T}}\mathcal{A}_i. \end{aligned}$$

Dynamics Equations in Closed Form

- Dynamic equations $\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta)$
- Definitions

$$\mathcal{V} = \begin{bmatrix} \mathcal{V}_1 \\ \vdots \\ \mathcal{V}_n \end{bmatrix} \in \mathbb{R}^{6n} \qquad \mathcal{F} = \begin{bmatrix} \mathcal{F}_1 \\ \vdots \\ \mathcal{F}_n \end{bmatrix} \in \mathbb{R}^{6n} \qquad \mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mathcal{A}_n \end{bmatrix} \in \mathbb{R}^{6n \times n}$$

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{G}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mathcal{G}_n \end{bmatrix} \in \mathbb{R}^{6n \times 6n} \qquad [\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\mathrm{ad}_{\mathcal{V}_1}] & 0 & \cdots & 0 \\ 0 & [\mathrm{ad}_{\mathcal{V}_2}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & [\mathrm{ad}_{\mathcal{V}_n}] \end{bmatrix} \in \mathbb{R}^{6n \times 6n}$$

$$[\mathrm{ad}_{\mathcal{A}\dot{\theta}}] = \begin{bmatrix} [\mathrm{ad}_{\mathcal{A}_1\dot{\theta}_1}] & 0 & \cdots & 0 \\ 0 & [\mathrm{ad}_{\mathcal{A}_2\dot{\theta}_2}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & [\mathrm{ad}_{\mathcal{A}_n\dot{\theta}_n}] \end{bmatrix} \in \mathbb{R}^{6n \times 6n} \qquad \mathcal{W}(\theta) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ [\mathrm{Ad}_{T_{21}}] & 0 & \cdots & 0 & 0 \\ 0 & [\mathrm{Ad}_{T_{32}}] & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & [\mathrm{Ad}_{T_{n,n-1}}] & 0 \end{bmatrix} \in \mathbb{R}^{6n \times 6n}$$

Dynamics Equations in Closed Form

$$\mathcal{V}_{\text{base}} = \begin{bmatrix} \mathrm{Ad}_{T_{10}}(\mathcal{V}_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n} \qquad \dot{\mathcal{V}}_{\text{base}} = \begin{bmatrix} \mathrm{Ad}_{T_{10}}(\dot{\mathcal{V}}_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n} \qquad \mathcal{F}_{\text{tip}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ad}^{\mathrm{T}}_{T_{n+1,n}}(\mathcal{F}_{n+1}) \end{bmatrix} \in \mathbb{R}^{6n}$$

Recursive inverse dynamics algorithm

$$\begin{aligned}
\mathcal{V} &= \mathcal{W}(\theta)\mathcal{V} + \mathcal{A}\dot{\theta} + \mathcal{V}_{\text{base}}, \\
\dot{\mathcal{V}} &= \mathcal{W}(\theta)\dot{\mathcal{V}} + \mathcal{A}\ddot{\theta} - [\mathrm{ad}_{\mathcal{A}\dot{\theta}}](\mathcal{W}(\theta)\mathcal{V} + \mathcal{V}_{\text{base}}) + \dot{\mathcal{V}}_{\text{base}}, \\
\mathcal{F} &= \mathcal{W}^{\mathrm{T}}(\theta)\mathcal{F} + \mathcal{G}\dot{\mathcal{V}} - [\mathrm{ad}_{\mathcal{V}}]^{\mathrm{T}}\mathcal{G}\mathcal{V} + \mathcal{F}_{\text{tip}}, \\
\tau &= \mathcal{A}^{\mathrm{T}}\mathcal{F}.
\end{aligned}$$

Dynamics Equations in Closed Form

- Define $\mathcal{L}(\theta) = (I - \mathcal{W}(\theta))^{-1}$

$$
\begin{aligned}
\mathcal{V} &= \mathcal{L}(\theta)\left(\mathcal{A}\dot{\theta} + \mathcal{V}_{\text{base}}\right), \\
\dot{\mathcal{V}} &= \mathcal{L}(\theta)\left(\mathcal{A}\ddot{\theta} + [\text{ad}_{\mathcal{A}\dot{\theta}}]\mathcal{W}(\theta)\mathcal{V} + [\text{ad}_{\mathcal{A}\dot{\theta}}]\mathcal{V}_{\text{base}} + \dot{\mathcal{V}}_{\text{base}}\right) \\
\mathcal{F} &= \mathcal{L}^T(\theta)\left(\mathcal{G}\dot{\mathcal{V}} - [\text{ad}_{\mathcal{V}}]^{\text{T}}\mathcal{G}\mathcal{V} + \mathcal{F}_{\text{tip}}\right), \\
\tau &= \mathcal{A}^T\mathcal{F}.
\end{aligned}
$$

Dynamics Equations in Closed Form

- If the robot applies an external wrench at the end-effector $\mathcal{F}_{\mathrm{tip}}$

End-effector torque $\tau = J^{\mathrm{T}}(\theta)\mathcal{F}_{\mathrm{tip}}$

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) + J^{\mathrm{T}}(\theta)\mathcal{F}_{\mathrm{tip}}$$

$$\begin{aligned}
M(\theta) &= \mathcal{A}^T \mathcal{L}^{\mathrm{T}}(\theta)\mathcal{G}\mathcal{L}(\theta)\mathcal{A}, \\
c(\theta, \dot{\theta}) &= -\mathcal{A}^T \mathcal{L}^T(\theta)\left(\mathcal{G}\mathcal{L}(\theta)\left[\mathrm{ad}_{\mathcal{A}\dot{\theta}}\right]\mathcal{W}(\theta) + \left[\mathrm{ad}_{\mathcal{V}}\right]^{\mathrm{T}}\mathcal{G}\right)\mathcal{L}(\theta)\mathcal{A}\dot{\theta}, \\
g(\theta) &= \mathcal{A}^T \mathcal{L}^T(\theta)\mathcal{G}\mathcal{L}(\theta)\dot{\mathcal{V}}_{\mathrm{base}}.
\end{aligned}$$

Robot Dynamics

- Study motion of robots with the forces and torques that cause them

- Equations of motion
  - A set of second-order differential equations

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

Joint variables $\theta \in \mathbb{R}^n$

Joint forces and torques $\tau \in \mathbb{R}^n$

$M(\theta) \in \mathbb{R}^{n \times n}$ a symmetric positive-definite **mass matrix**

$h(\theta, \dot{\theta}) \in \mathbb{R}^n$ forces that lump together centripetal, Coriolis, gravity, and friction terms that depend on θ and $\dot{\theta}$

Forward Dynamics of Open Chains

- Forward dynamics $M(\theta)\ddot{\theta} = \tau(t) - h(\theta, \dot{\theta}) - J^{\mathrm{T}}(\theta)\mathcal{F}_{\mathrm{tip}}$
  - Given $\theta, \ \dot{\theta}, \ \tau \ \mathcal{F}_{\mathrm{tip}}$ Solve $\ddot{\theta}$

- $h(\theta, \dot{\theta})$ can be computed by the inverse dynamics algorithm with $\ddot{\theta} = 0$ and $\mathcal{F}_{\mathrm{tip}} = 0$

- We can solve

$$M\ddot{\theta} = b, \text{ for } \ddot{\theta}$$

$$b = \tau(t) - h(\theta, \dot{\theta}) - J^{\mathrm{T}}(\theta)\mathcal{F}_{\mathrm{tip}}$$

Forward Dynamics of Open Chains

- Simulate the motion of a robot

$$\ddot{\theta} = ForwardDynamics(\theta, \dot{\theta}, \tau, \mathcal{F}_{\text{tip}})$$

First-order differential equations

$$q_1 = \theta, \; q_2 = \dot{\theta}$$

$$\begin{aligned} \dot{q}_1 &= q_2, \\ \dot{q}_2 &= ForwardDynamics(q_1, q_2, \tau, \mathcal{F}_{\text{tip}}) \end{aligned}$$

First-order Euler iteration

$$\begin{aligned} q_1(t + \delta t) &= q_1(t) + q_2(t)\delta t, \\ q_2(t + \delta t) &= q_2(t) + ForwardDynamics(q_1, q_2, \tau, \mathcal{F}_{\text{tip}})\delta t \end{aligned}$$

Initial values $q_1(0) = \theta(0)$ and $q_2(0) = \dot{\theta}(0)$

Summary

- Newton-Euler Inverse Dynamics Algorithm
- Forward Dynamics of Open Chains

Further Reading

- Chapter 8 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.

- Dynamics of Dynamics of Open Chains: Newton Euler Approach. Prof. Wei Zhang, Southern University of Science and Technology, Shenzhen, China https://www.wzhanglab.site/wp-content/uploads/2024/06/LN9_DynamicsOfOpenChain.pdf