Dynamics of a Single Rigid Body

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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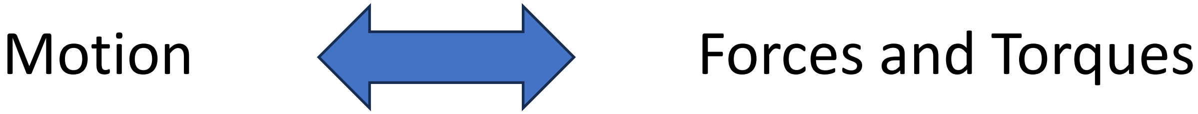
Robot Dynamics

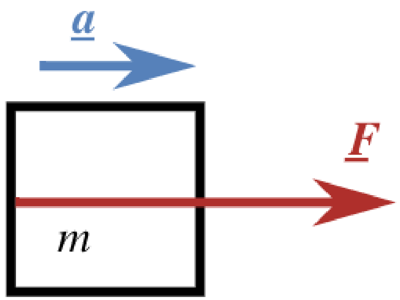
- Study motion of robots with the forces and torques that cause them



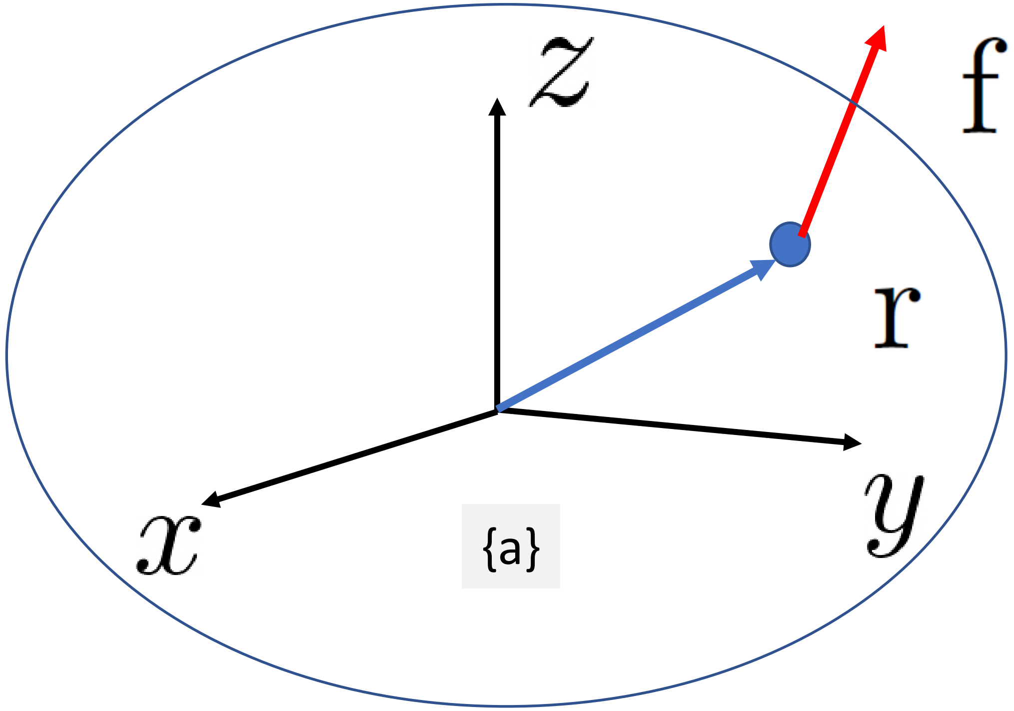
Motion

Forces and Torques

- Using Newton’s second law F = ma



Torque



A rigid body

Point $r_a \in \mathbb{R}^3$

Force $f_a \in \mathbb{R}^3$

Torque or Moment

$$m_a \in \mathbb{R}^3$$

$$m_a = r_a \times f_a$$

Spatial Force or Wrench

- Merge moment and force in frame {a}

Wrench $$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix} \in \mathbb{R}^6$$

- If more than one wrenches act on a rigid body, the total wrench is the vector sum of the wrenches

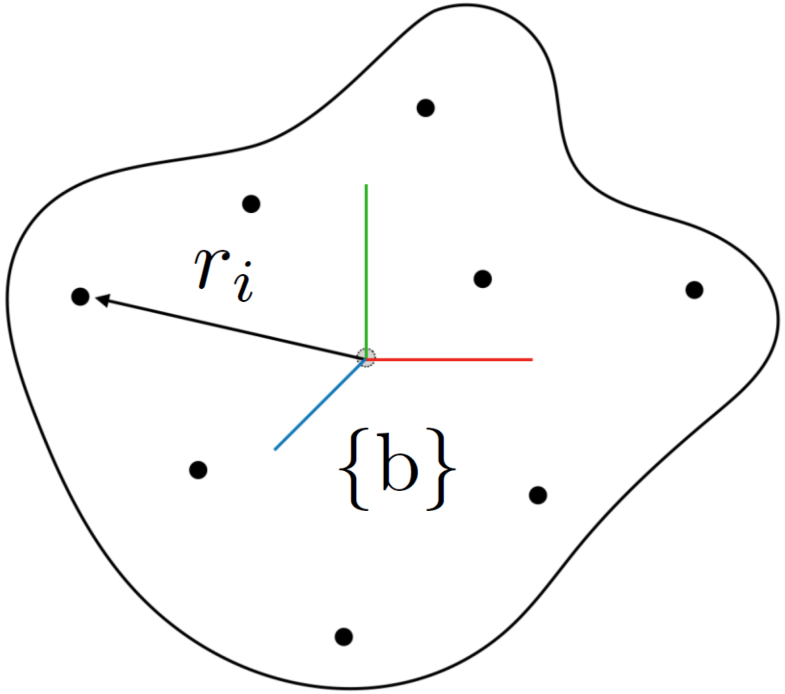
- Pure moment: a wrench with a zero linear component

Dynamics of a Single Rigid Body

- A rigid body with a set of point masses
- Total mass $\mathfrak{m} = \sum_i \mathfrak{m}_i$
- The origin of the body frame

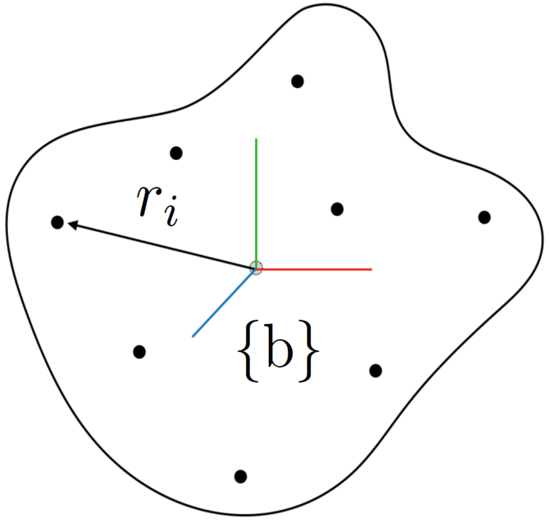
  Center of mass $\sum_i \mathfrak{m}_i r_i = 0$

- If some other point is chosen as origin, move the origin to $(1/\mathfrak{m}) \sum_i \mathfrak{m}_i r_i$



Dynamics of a Single Rigid Body

- Assume the body is moving with a body twist $\mathcal{V}_b = (\omega_b, v_b)$
- $p_i(t)$ be the time-varying position of $\mathfrak{m}_i$, initially at r_i



$$\dot{p}_i = v_b + \omega_b \times p_i$$

$$\ddot{p}_i = \dot{v}_b + \frac{d}{dt}\omega_b \times p_i + \omega_b \times \frac{d}{dt}p_i$$

$$= \dot{v}_b + \dot{\omega}_b \times p_i + \omega_b \times (v_b + \omega_b \times p_i)$$

$$\ddot{p}_i = \dot{v}_b + [\dot{\omega}_b]r_i + [\omega_b]v_b + [\omega_b]^2 r_i$$

Dynamics of a Single Rigid Body

- For a point mass $f_i = \mathfrak{m}_i \ddot{p}_i$

$$f_i = \mathfrak{m}_i(\dot{v}_b + [\dot{\omega}_b]r_i + [\omega_b]v_b + [\omega_b]^2 r_i)$$

- Moment of the point mass $m_i = [r_i]f_i$
- Total force and moment on the body

Wrench $\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \sum_i m_i \\ \sum_i f_i \end{bmatrix}$

Dynamics of a Single Rigid Body

- Linear dynamics

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$$\textstyle\sum_i \mathfrak{m}_i [r_i] = 0$$

$$\begin{aligned} f_b &= \sum_i \mathfrak{m}_i(\dot{v}_b + [\dot{\omega}_b] r_i + [\omega_b] v_b + [\omega_b]^2 r_i) \\ &= \sum_i \mathfrak{m}_i(\dot{v}_b + [\omega_b] v_b) - \cancelto{0}{\sum_i \mathfrak{m}_i [r_i] \dot{\omega}_b} + \cancelto{0}{\sum_i \mathfrak{m}_i [r_i][\omega_b] \omega_b} \\ &= \sum_i \mathfrak{m}_i(\dot{v}_b + [\omega_b] v_b) \\ &= \mathfrak{m}(\dot{v}_b + [\omega_b] v_b). \end{aligned}$$

Dynamics of a Single Rigid Body

$$[a] = -[a]^{\mathrm{T}}$$

$$[a]b = -[b]a$$

$$[a][b] = ([b][a])^{\mathrm{T}}$$

- Rotational dynamics

$$\begin{aligned}
m_b &= \sum_i \mathfrak{m}_i [r_i](\dot{v}_b + [\dot{\omega}_b] r_i + [\omega_b] v_b + [\omega_b]^2 r_i) \\
&= \cancelto{0}{\sum_i \mathfrak{m}_i [r_i] \dot{v}_b} + \cancelto{0}{\sum_i \mathfrak{m}_i [r_i][\omega_b] v_b} \\
&\quad + \sum_i \mathfrak{m}_i [r_i]([\dot{\omega}_b] r_i + [\omega_b]^2 r_i) \\
&= \sum_i \mathfrak{m}_i (-[r_i]^2 \dot{\omega}_b - [r_i][\omega_b][r_i]\omega_b) \\
&= \sum_i \mathfrak{m}_i \left(-[r_i]^2 \dot{\omega}_b - [\omega_b][r_i]^2 \omega_b\right) \\
&= \left(-\sum_i \mathfrak{m}_i [r_i]^2\right) \dot{\omega}_b + [\omega_b] \left(-\sum_i \mathfrak{m}_i [r_i]^2\right) \omega_b \\
&= \mathcal{I}_b \dot{\omega}_b + [\omega_b] \mathcal{I}_b \omega_b,
\end{aligned}$$

Fact $[r_i \times \omega_b] = [r_i][\omega_b] - [\omega_b][r_i]$

Body's rotational inertia matrix

$$\mathcal{I}_b = -\sum\nolimits_i \mathfrak{m}_i [r_i]^2 \in \mathbb{R}^{3\times 3}$$

symmetric and positive definite

Euler's equation for a rotating rigid body

Dynamics of a Single Rigid Body

Body twist $\mathcal{V}_b = (\omega_b, v_b)$

- Linear dynamics

$$f_b = \mathfrak{m}(\dot{v}_b + [\omega_b]v_b)$$

- Rotational dynamics

$$m_b \quad = \quad \mathcal{I}_b\dot{\omega}_b + [\omega_b]\mathcal{I}_b\omega_b$$

$$\mathcal{I}_b = -\sum_i \mathfrak{m}_i [r_i]^2 \in \mathbb{R}^{3\times 3}$$

- Rotational kinetic energy

$$\mathcal{K} = \frac{1}{2}\omega_b^{\mathrm{T}}\mathcal{I}_b\omega_b$$

Dynamics of a Single Rigid Body

- Rotational inertia matrix $\mathcal{I}_b = -\sum_i \mathfrak{m}_i [r_i]^2 \in \mathbb{R}^{3\times 3}$

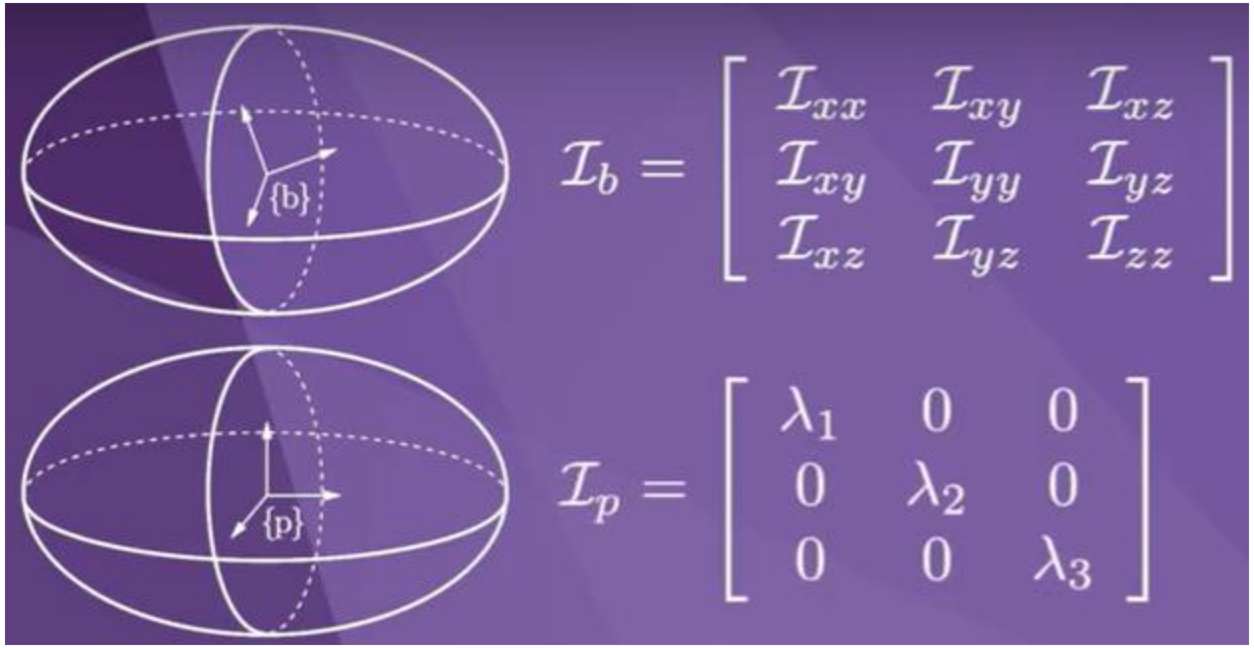
$$\begin{aligned}\mathcal{I}_b &= \begin{bmatrix} \sum \mathfrak{m}_i(y_i^2+z_i^2) & -\sum \mathfrak{m}_i x_i y_i & -\sum \mathfrak{m}_i x_i z_i \\ -\sum \mathfrak{m}_i x_i y_i & \sum \mathfrak{m}_i(x_i^2+z_i^2) & -\sum \mathfrak{m}_i y_i z_i \\ -\sum \mathfrak{m}_i x_i z_i & -\sum \mathfrak{m}_i y_i z_i & \sum \mathfrak{m}_i(x_i^2+y_i^2) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}.\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{xx} &= \int_{\mathcal{B}} (y^2+z^2)\rho(x,y,z)\,dV & \mathcal{I}_{xy} &= -\int_{\mathcal{B}} xy\rho(x,y,z)\,dV \\ \mathcal{I}_{yy} &= \int_{\mathcal{B}} (x^2+z^2)\rho(x,y,z)\,dV & \mathcal{I}_{xz} &= -\int_{\mathcal{B}} xz\rho(x,y,z)\,dV \\ \mathcal{I}_{zz} &= \int_{\mathcal{B}} (x^2+y^2)\rho(x,y,z)\,dV & \mathcal{I}_{yz} &= -\int_{\mathcal{B}} yz\rho(x,y,z)\,dV.\end{aligned}$$

mass density function $\rho(x, y, z)$

Inertia Matrix

- Principal axes of inertia: eigenvectors of $\mathcal{I}_b$
  - Directions given by eigenvectors
  - Eigenvalues are principal moments of inertia



$$\mathcal{I}_b = \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}$$

$$\mathcal{I}_p = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Inertia Matrix

- General rotation dynamics

$$m_b = \mathcal{I}_b \dot{\omega}_b + [\omega_b] \mathcal{I}_b \omega_b$$

- If the principal axes are aligned with the axes of {b}, $\mathcal{I}_b$ is a diagonal matrix

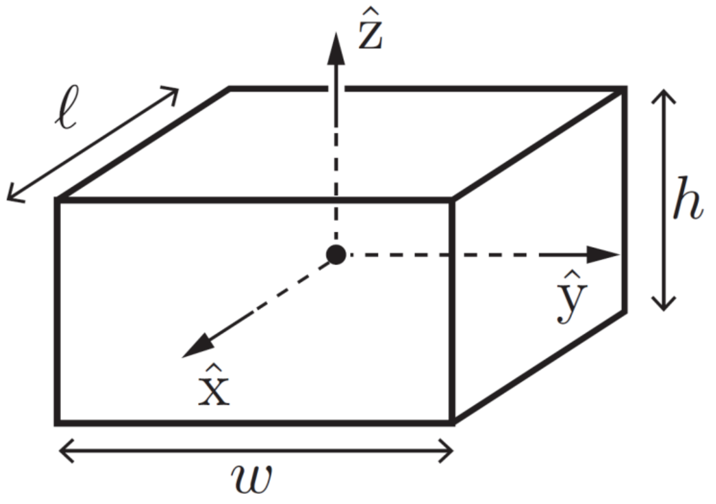
rotational dynamics

$$m_b = \begin{bmatrix} \mathcal{I}_{xx}\dot{\omega}_x + (\mathcal{I}_{zz} - \mathcal{I}_{yy})\omega_y\omega_z \\ \mathcal{I}_{yy}\dot{\omega}_y + (\mathcal{I}_{xx} - \mathcal{I}_{zz})\omega_x\omega_z \\ \mathcal{I}_{zz}\dot{\omega}_z + (\mathcal{I}_{yy} - \mathcal{I}_{xx})\omega_x\omega_y \end{bmatrix}$$

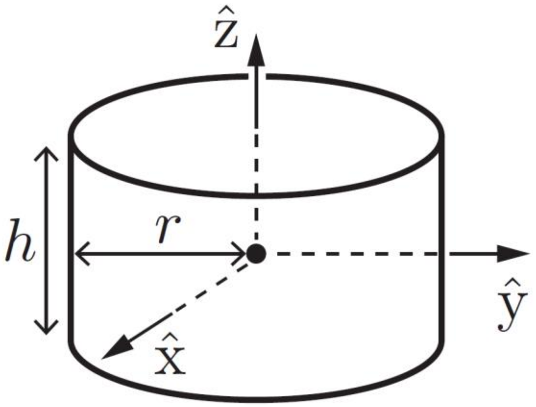
$$\omega_b = (\omega_x, \omega_y, \omega_z)$$

Inertia Matrix

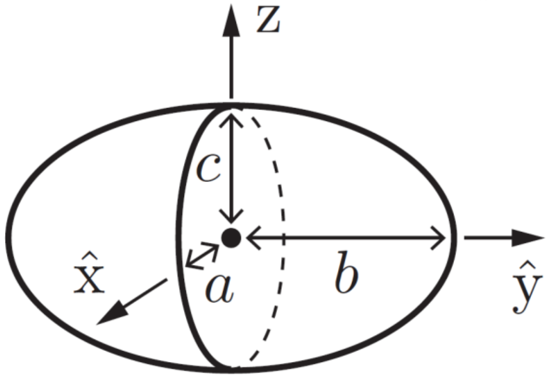
$$\begin{aligned}
\mathcal{I}_{xx} &= \int_{\mathcal{B}} (y^2 + z^2)\rho(x,y,z)\, dV \\
\mathcal{I}_{yy} &= \int_{\mathcal{B}} (x^2 + z^2)\rho(x,y,z)\, dV \\
\mathcal{I}_{zz} &= \int_{\mathcal{B}} (x^2 + y^2)\rho(x,y,z)\, dV
\end{aligned}$$



rectangular parallelepiped:
volume $= abc$,
$\mathcal{I}_{xx} = \mathfrak{m}(w^2 + h^2)/12$,
$\mathcal{I}_{yy} = \mathfrak{m}(\ell^2 + h^2)/12$,
$\mathcal{I}_{zz} = \mathfrak{m}(\ell^2 + w^2)/12$



circular cylinder:
volume $= \pi r^2 h$,
$\mathcal{I}_{xx} = \mathfrak{m}(3r^2 + h^2)/12$,
$\mathcal{I}_{yy} = \mathfrak{m}(3r^2 + h^2)/12$,
$\mathcal{I}_{zz} = \mathfrak{m}r^2/2$



ellipsoid:
volume $= 4\pi abc/3$,
$\mathcal{I}_{xx} = \mathfrak{m}(b^2 + c^2)/5$,
$\mathcal{I}_{yy} = \mathfrak{m}(a^2 + c^2)/5$,
$\mathcal{I}_{zz} = \mathfrak{m}(a^2 + b^2)/5$

Inertia Matrix

- Inertia matrix in a rotated frame {c}
- Kinetic energy is the same in different frame

$$\begin{aligned}\frac{1}{2}\omega_c^{\mathrm{T}}\mathcal{I}_c\omega_c &= \frac{1}{2}\omega_b^{\mathrm{T}}\mathcal{I}_b\omega_b \\ &= \frac{1}{2}(R_{bc}\omega_c)^{\mathrm{T}}\mathcal{I}_b(R_{bc}\omega_c) \\ &= \frac{1}{2}\omega_c^{\mathrm{T}}(R_{bc}^{\mathrm{T}}\mathcal{I}_b R_{bc})\omega_c.\end{aligned}$$

$$\mathcal{I}_c = R_{bc}^{\mathrm{T}}\mathcal{I}_b R_{bc}$$

Steiner's theorem

- The inertia matrix $\mathcal{I}_q$ about a frame aligned with {b}, but at a point in {b} $q = (q_x, q_y, q_z)$, is related to the inertia matrix calculated at the center of mass by

$$\mathcal{I}_q = \mathcal{I}_b + \mathfrak{m}(q^{\mathrm{T}} q I - q q^{\mathrm{T}})$$

- Parallel-axis theorem: the scalar inertia $\mathcal{I}_d$ about an axis parallel to, but a distance d from, an axis through the center of mass is

$$\mathcal{I}_d = \mathcal{I}_{\mathrm{cm}} + \mathfrak{m} d^2$$

Inertia Matrix

- Change of reference frame

$$\mathcal{I}_c = R_{bc}^{\mathrm{T}} \mathcal{I}_b R_{bc}$$

$$\mathcal{I}_q = \mathcal{I}_b + \mathfrak{m}(q^{\mathrm{T}} q I - q q^{\mathrm{T}})$$

Further Reading

- Chapter 8 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.

- Dynamics of a Single Rigid Body. Prof. Wei Zhang, Southern University of Science and Technology, Shenzhen https://www2.ece.ohio-state.edu/~zhang/RoboticsClass/docs/LN11_RigidBodyDynamics_a.pdf