Velocity Kinematics I

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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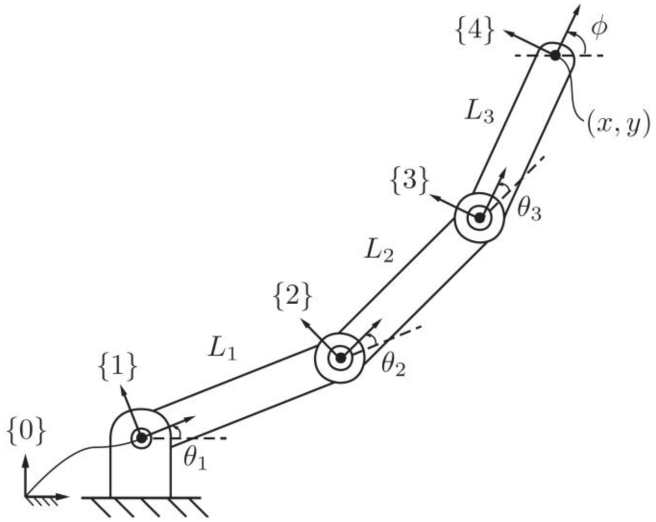
Forward Kinematics

- Forward kinematics of a robot: calculation of the position and orientation of its end-effector from its joint coordinates

Joint coordinates θ → End-effector transformation

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Forward Kinematics



- Method 1: uses homogeneous transformations
  - Need to define the coordinates of frames
  - Denavit-Hartenberg Parameters

$$T_{04} = T_{01}T_{12}T_{23}T_{34}$$

- Method 2: uses screw-axis representations of transformations
  - No need to define frame references

Space form $$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

Body form $$T_{04} = M e^{[\mathcal{B}]_1\theta_1} e^{[\mathcal{B}]_2\theta_2} e^{[\mathcal{B}]_3\theta_3}$$

Velocity Kinematics

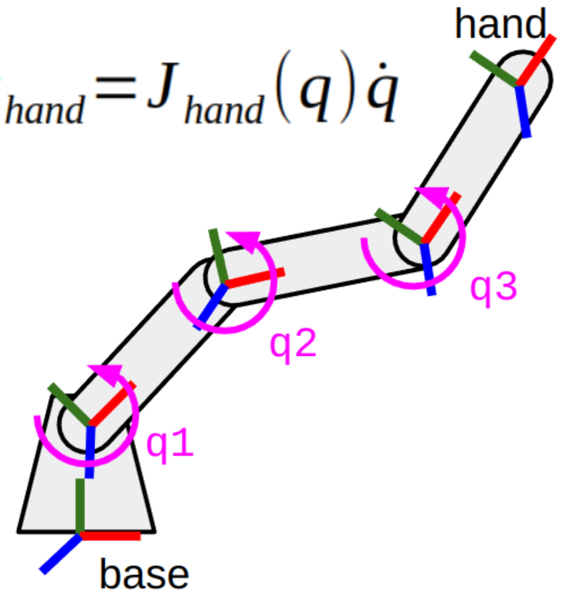
- Given joint positions and velocities $\theta \in \mathbb{R}^n \qquad \dot{\theta}$

- Compute the twist of the end-effector
  - Angular velocity and linear velocity

$$\theta \quad \dot{\theta} \quad \Rightarrow \quad \mathbf{v}_{\text{hand}}$$

End-effector velocity

$$v_{hand} = J_{hand}(q)\dot{q}$$



Jacobian

- Assume end-effector configuration $x \in \mathbb{R}^m$
- End-effector velocity $\dot{x} = dx/dt \in \mathbb{R}^m$
- Forward kinematics $x(t) = f(\theta(t))$ $\theta \in \mathbb{R}^n$ Joint variable
- Chain rule

$$\dot{x} = \frac{\partial f(\theta)}{\partial \theta} \frac{d\theta(t)}{dt} = \frac{\partial f(\theta)}{\partial \theta} \dot{\theta}$$
$$= J(\theta)\dot{\theta},$$

$J(\theta) \in \mathbb{R}^{m \times n}$ Jacobian

$\dot{\theta}$ Joint velocity

Gradients

How to compute gradient?

$L(\mathbf{y})$ scalar $\mathbf{y} : m \times 1$

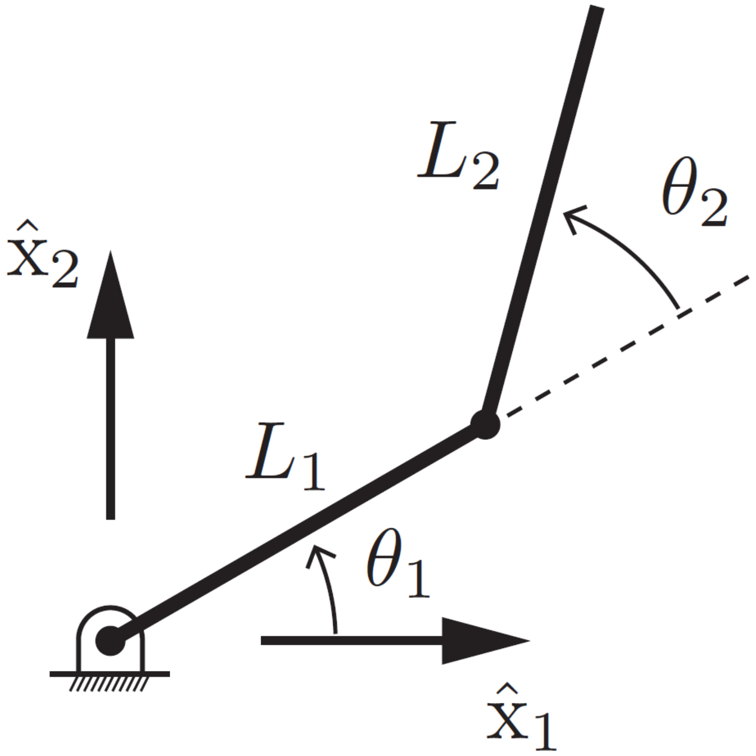
$$\frac{\partial L}{\partial \mathbf{y}} \begin{bmatrix} \frac{\partial L}{y_1} & \frac{\partial L}{y_2} & \dots & \frac{\partial L}{y_m} \end{bmatrix}$$

$$1 \times m$$

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \nabla f_1(\mathbf{x}) \\ \nabla f_2(\mathbf{x}) \\ \dots \\ \nabla f_m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \mathbf{x}} f_1(\mathbf{x}) \\ \frac{\partial}{\partial \mathbf{x}} f_2(\mathbf{x}) \\ \dots \\ \frac{\partial}{\partial \mathbf{x}} f_m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(\mathbf{x}) & \frac{\partial}{\partial x_2} f_1(\mathbf{x}) & \dots & \frac{\partial}{\partial x_n} f_1(\mathbf{x}) \\ \frac{\partial}{\partial x_1} f_2(\mathbf{x}) & \frac{\partial}{\partial x_2} f_2(\mathbf{x}) & \dots & \frac{\partial}{\partial x_n} f_2(\mathbf{x}) \\ & & \dots & \\ \frac{\partial}{\partial x_1} f_m(\mathbf{x}) & \frac{\partial}{\partial x_2} f_m(\mathbf{x}) & \dots & \frac{\partial}{\partial x_n} f_m(\mathbf{x}) \end{bmatrix}$$

Jacobian matrix

Jacobian



a 2R planar open chain

Forward kinematics

$$\begin{aligned} x_1 &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ x_2 &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2). \end{aligned}$$

Differentiate with respect to time

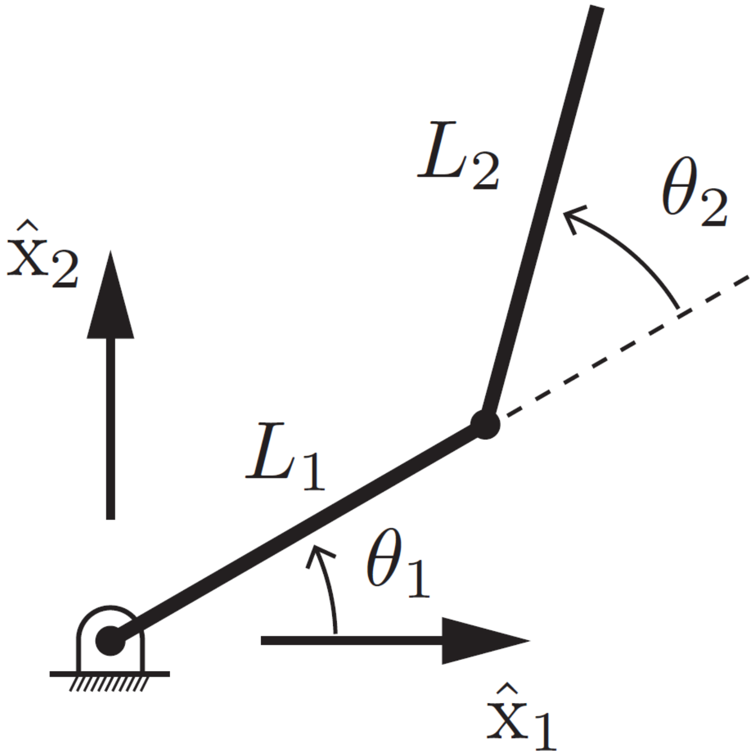
$$\begin{aligned} \dot{x}_1 &= -L_1\dot{\theta}_1 \sin\theta_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2) \\ \dot{x}_2 &= L_1\dot{\theta}_1 \cos\theta_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2), \end{aligned}$$

$$\dot{x} = J(\theta)\dot{\theta}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$v_{\text{tip}} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

Jacobian



a 2R planar open chain

$$v_{\text{tip}} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

$J_1(\theta)$ and $J_2(\theta)$ Are not colinear, v can be any direction in the x-y plane

$J_1(\theta)$ and $J_2(\theta)$ Depends on theta

θ_2 is 0° or 180°

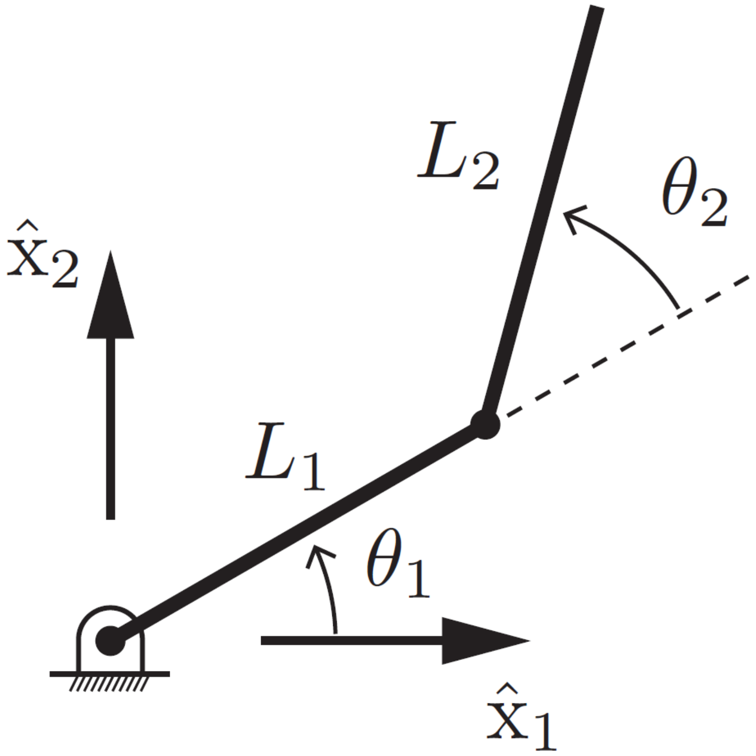
$J_1(\theta)$ and $J_2(\theta)$ Are colinear

Singularities: where the robot tip is unable to generate velocities in certain directions.

Jacobian

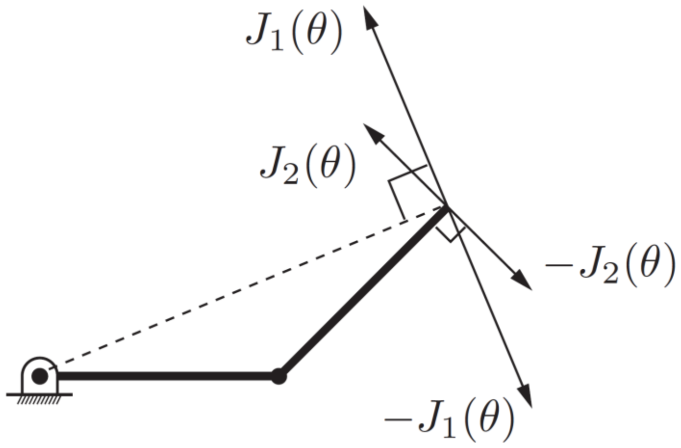
$$v_{\text{tip}} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

$$L_1 = L_2 = 1$$



a 2R planar open chain

$$\theta = (0, \pi/4) \quad J\left(\begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}\right) = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix}$$

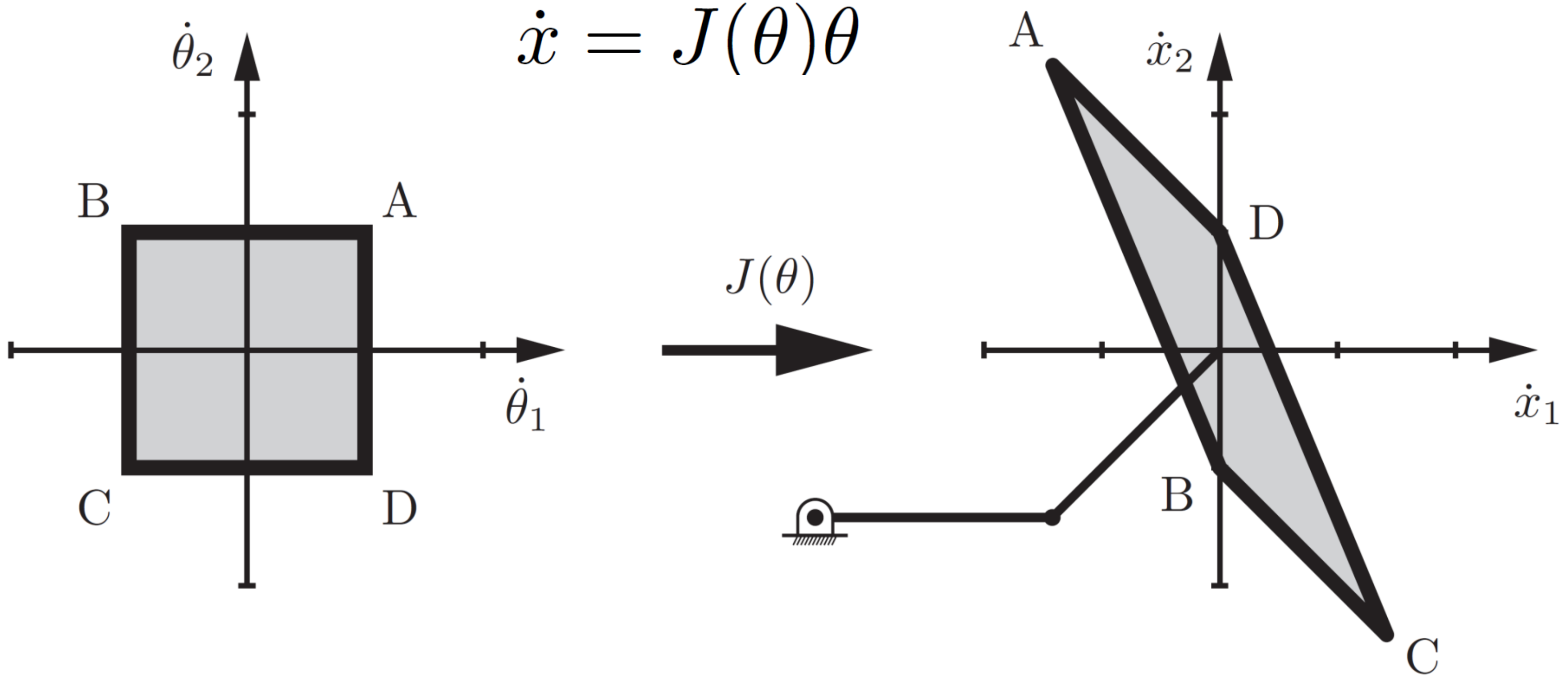


$$\theta = (0, 3\pi/4) \quad J\left(\begin{bmatrix} 0 \\ 3\pi/4 \end{bmatrix}\right) = \begin{bmatrix} -0.71 & -0.71 \\ 0.29 & -0.71 \end{bmatrix}$$

Jacobian

- Mapping of speed

$$\dot{x} = J(\theta)\dot{\theta}$$



Jacobian

- Mapping of speed

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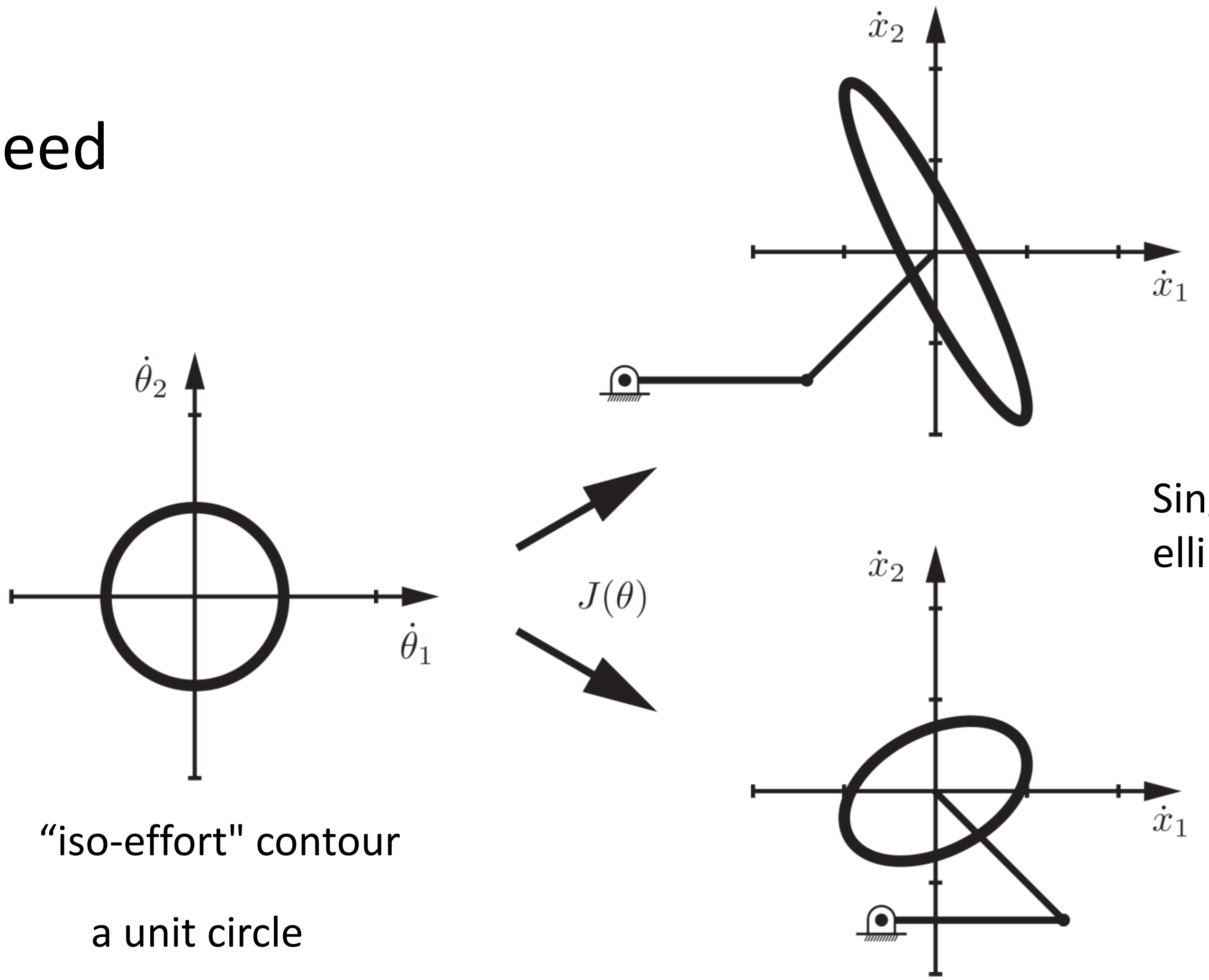

“iso-effort" contour

a unit circle

Singularity: manipulability ellipsoid becomes a line

manipulability ellipsoid

Recall Twists

- Spatial twist (spatial velocity in the space frame)

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6 \qquad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3)$$

- Relationship

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}[\mathcal{V}_s]T \end{aligned} \qquad [\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}$$

Manipulator Jacobian

- Forward kinematics

$$T(\theta_1, \ldots, \theta_n) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \ldots e^{[\mathcal{S}_n]\theta_n} M$$

$$[\mathcal{V}_s] = \dot{T}T^{-1}$$

$$\begin{aligned} \dot{T} &= \left(\frac{d}{dt} e^{[\mathcal{S}_1]\theta_1}\right) \cdots e^{[\mathcal{S}_n]\theta_n} M + e^{[\mathcal{S}_1]\theta_1} \left(\frac{d}{dt} e^{[\mathcal{S}_2]\theta_2}\right) \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots \\ &= [\mathcal{S}_1]\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2]\dot{\theta}_2 e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots \end{aligned}$$

$\dot{\theta}_i$ is a scalar

$$T^{-1} = M^{-1} e^{-[\mathcal{S}_n]\theta_n} \ldots e^{-[\mathcal{S}_1]\theta_1}$$

$$d(e^{A\theta})/dt = Ae^{A\theta}\dot{\theta} = e^{A\theta}A\dot{\theta}$$

Proposition 3.10

Space Jacobian

Adjoint map associated with T

$$\mathcal{V}' = \mathrm{Ad}_T(\mathcal{V})$$

$$[\mathcal{V}'] = T[\mathcal{V}]T^{-1}$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$[\mathcal{V}_s] = \dot{T}T^{-1}$$

$$[\mathcal{V}_s] = [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_2 + e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}[\mathcal{S}_3]e^{-[\mathcal{S}_2]\theta_2}e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_3 + \cdots$$

Adjoint mapping

$$\mathcal{V}_s = \underbrace{\mathcal{S}_1}_{J_{s1}}\dot{\theta}_1 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}}\dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}}\dot{\theta}_3 + \cdots$$

$$\mathcal{V}_s = J_{s1}\dot{\theta}_1 + J_{s2}(\theta)\dot{\theta}_2 + \cdots + J_{sn}(\theta)\dot{\theta}_n$$

Summary

- Velocity kinematics
- Jacobian
  - Space Jacobian

Further Reading

- Chapter 5 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.
- T. Yoshikawa. Manipulability of robotic mechanisms. International Journal of Robotics Research, 4(2):3-9, 1985.