Product of Exponentials Formula and URDF

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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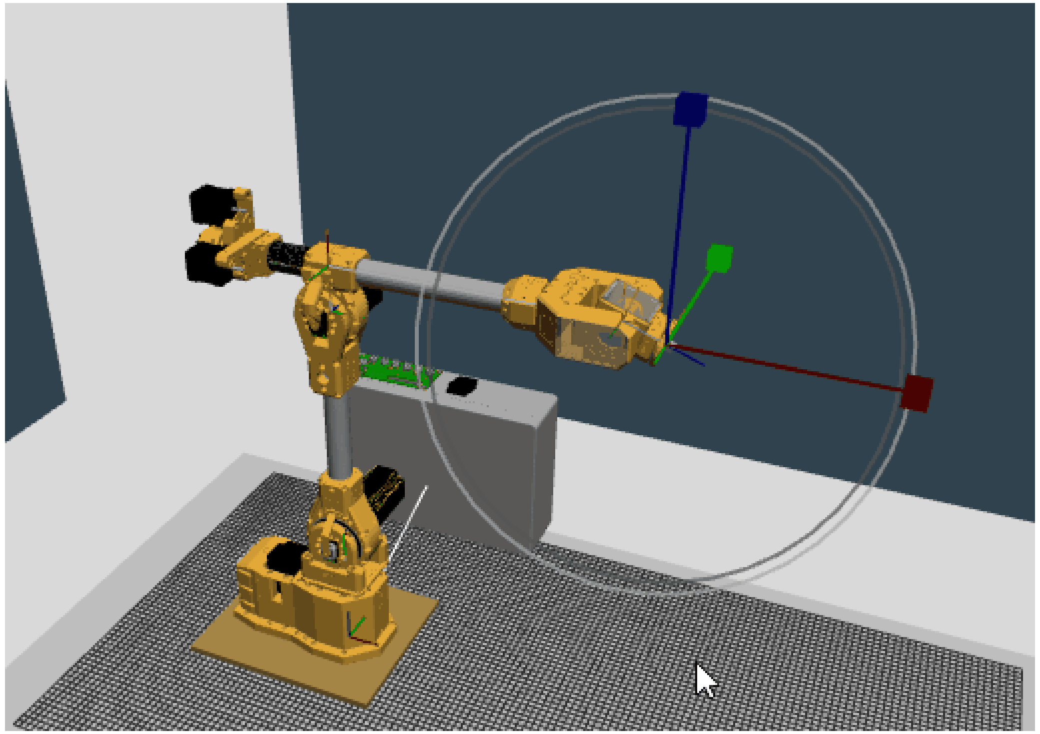
Robot Kinematics

- The relationship between a robot's joint coordinates and its spatial layout



https://www.marginallyclever.com/2020/04/gradient-descent-inverse-kinematics-for-6dof-robot-arms/

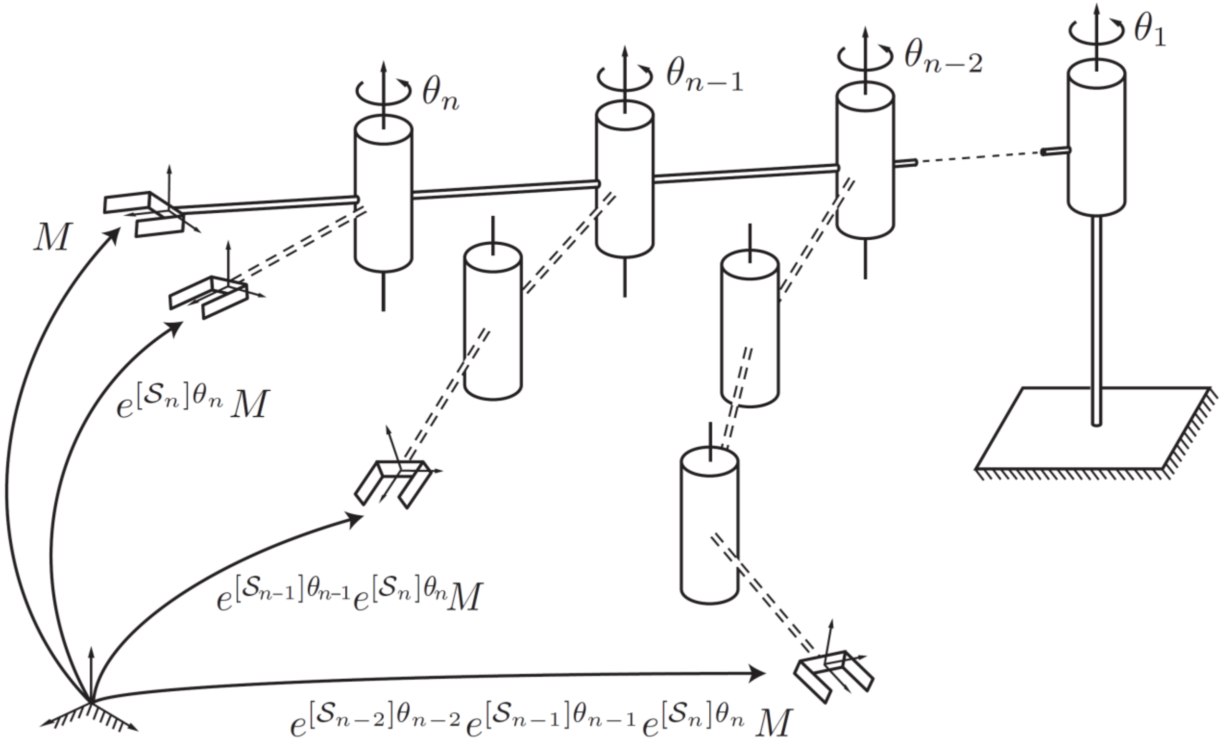
Forward Kinematics

- Forward kinematics of a robot: calculation of the position and orientation of its end-effector from its joint coordinates

Joint coordinates θ → End-effector transformation

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Product of Exponentials Formula



- Each link apply a screw motion to all the outward links
- Base frame {s}
- End-effector frame {b}

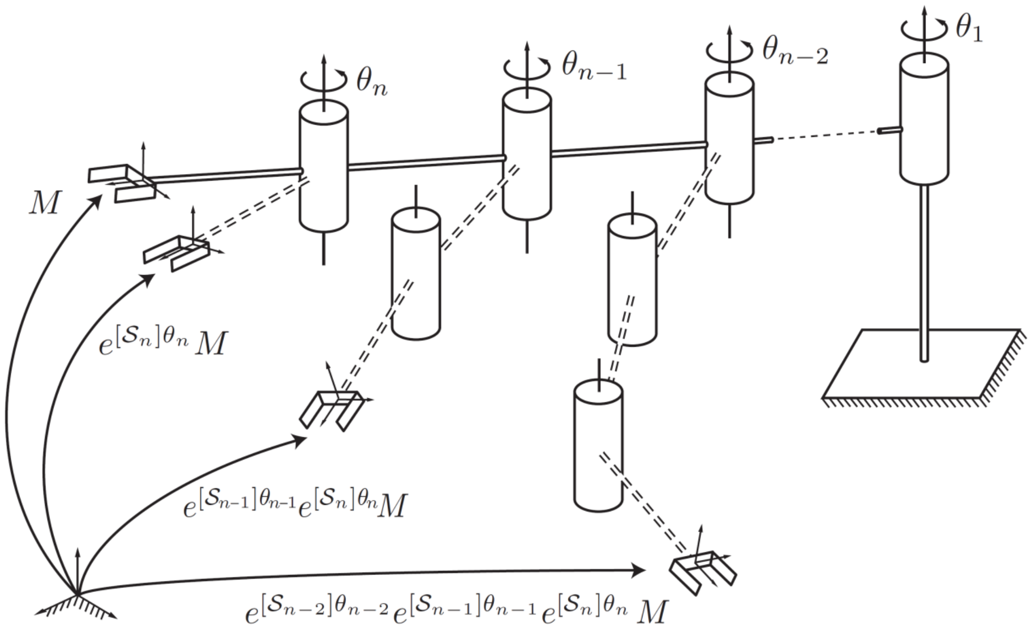
$$M \in SE(3)$$

{b} in {s} when all the joint values are zeros

$$T = e^{[\mathcal{S}_n]\theta_n} M$$

{b} in {s} when joint n with value θ_n

Product of Exponentials Formula



$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Joint values $(\theta_1, \ldots, \theta_n)$

- Space form of the product of exponentials formula
- Unlike D-H representation, no link reference frames need to be defined

Exponential Coordinates of Rigid-Body Motions

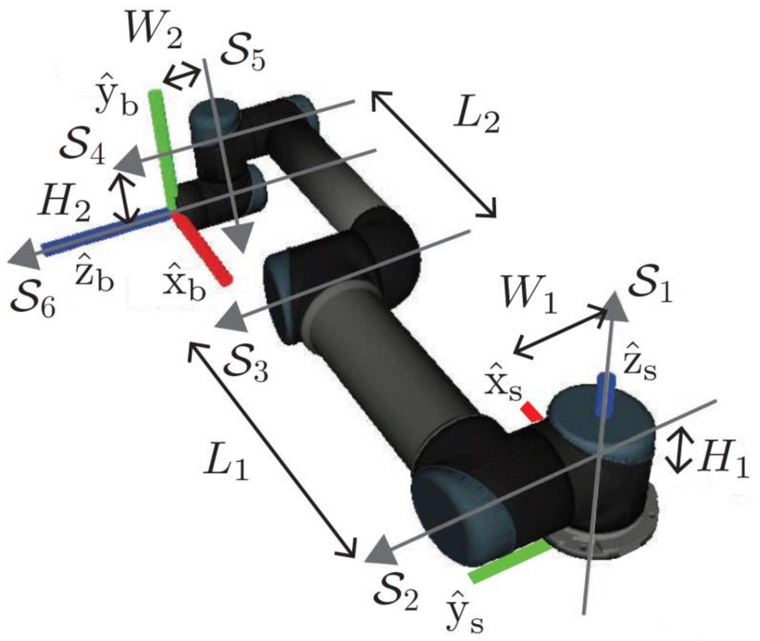
$$T(t) = e^{[\mathcal{S}]t}$$

If $\|\omega\| = 1$ $\quad e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}$

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \in SO(3)$$

If $\omega = 0$ and $\|v\| = 1$ $\quad e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$

Product of Exponentials Formula



Universal Robots' UR5 6R robot arm

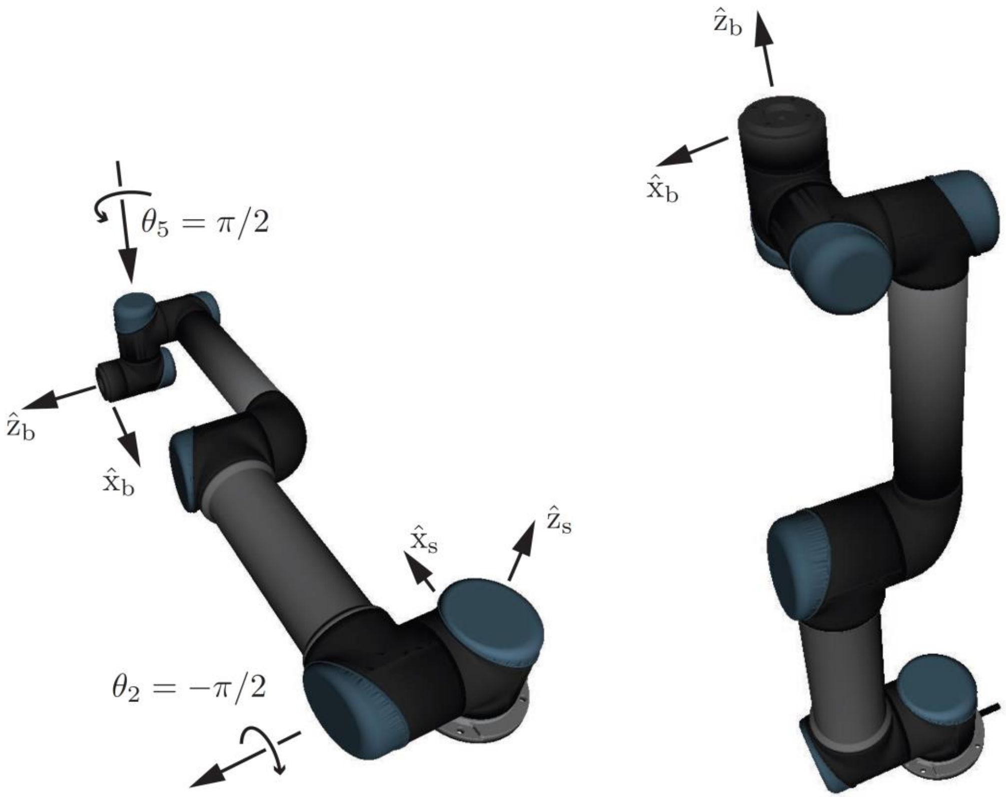
$$W_1 = 109 \text{ mm}, \ W_2 = 82 \text{ mm}, \ L_1 = 425 \text{ mm}$$

$$L_2 = 392 \text{ mm}, \ H_1 = 89 \text{ mm} \quad H_2 = 95 \text{ mm}$$

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1,0,0)$
3	$(0,1,0)$	$(-H_1,0,L_1)$
4	$(0,1,0)$	$(-H_1,0,L_1+L_2)$
5	$(0,0,-1)$	$(-W_1,L_1+L_2,0)$
6	$(0,1,0)$	$(H_2-H_1,0,L_1+L_2)$

Product of Exponentials Formula



Universal Robots' UR5 6R robot arm

$W_1 = 109$ mm, $W_2 = 82$ mm, $L_1 = 425$ mm

$L_2 = 392$ mm, $H_1 = 89$ mm $H_2 = 95$ mm

$$\theta_2 = -\pi/2 \text{ and } \theta_5 = \pi/2.$$

$$\begin{aligned} T(\theta) &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M \\ &= I e^{-[\mathcal{S}_2]\pi/2} I^2 e^{[\mathcal{S}_5]\pi/2} I M \\ &= e^{-[\mathcal{S}_2]\pi/2} e^{[\mathcal{S}_5]\pi/2} M \end{aligned}$$

$$e^{-[\mathcal{S}_2]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.089 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.089 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{[\mathcal{S}_5]\pi/2} = \begin{bmatrix} 0 & 1 & 0 & 0.708 \\ -1 & 0 & 0 & 0.926 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T(\theta) = e^{-[\mathcal{S}_2]\pi/2} e^{[\mathcal{S}_5]\pi/2} M = \begin{bmatrix} 0 & -1 & 0 & 0.095 \\ 1 & 0 & 0 & 0.109 \\ 0 & 0 & 1 & 0.988 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall Twists

- Spatial twist (spatial velocity in the space frame)

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6 \qquad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3)$$

- Relationship

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}[\mathcal{V}_s]T \end{aligned} \qquad [\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}$$

Recall Adjoint Representations

- The adjoint representation of $T = (R, p) \in SE(3)$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- The adjoint map associated with T

$$\mathcal{V} \in \mathbb{R}^6 \qquad \mathcal{V}' = [\mathrm{Ad}_T]\mathcal{V} \quad \text{or} \quad \mathcal{V}' = \mathrm{Ad}_T(\mathcal{V})$$

$$[\mathcal{V}] \in se(3) \qquad [\mathcal{V}'] = T[\mathcal{V}]T^{-1}$$

Recall Twists

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s$$

In general

$$\mathcal{V}_c = [\mathrm{Ad}_{T_{cd}}]\mathcal{V}_d, \qquad \mathcal{V}_d = [\mathrm{Ad}_{T_{dc}}]\mathcal{V}_c$$

Screw Axes in the End-Effector Frame

- Proposition $e^{M^{-1}PM} = M^{-1}e^{P}M \qquad Me^{M^{-1}PM} = e^{P}M$
- PoE formula

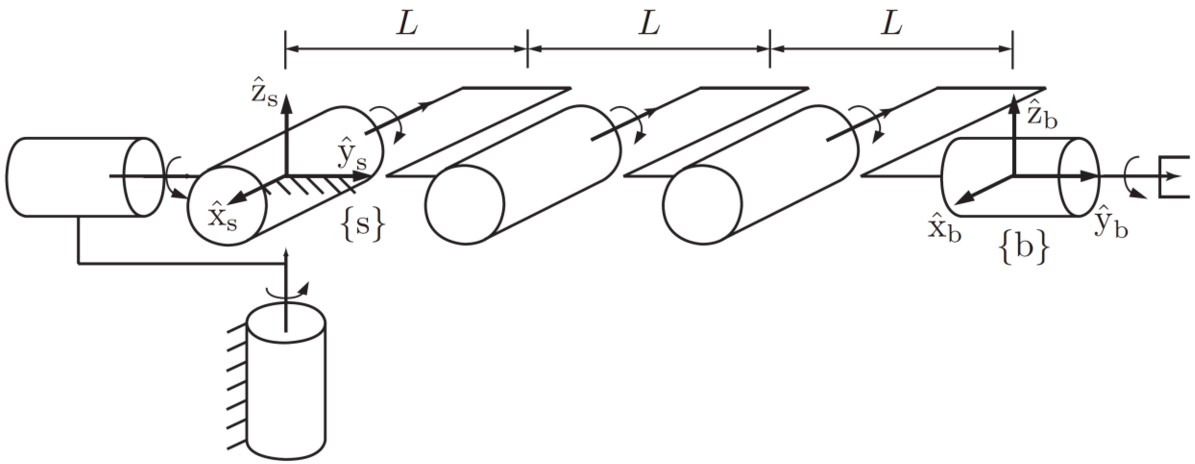
$$[\mathcal{B}_i] = M^{-1}[\mathcal{S}_i]M$$

$$\mathcal{B}_i = [\mathrm{Ad}_{M^{-1}}]\mathcal{S}_i, \; i = 1, \ldots, n$$

$$\begin{aligned}
T(\theta) &= e^{[\mathcal{S}_1]\theta_1} \cdots \boxed{e^{[\mathcal{S}_n]\theta_n} M} \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= M e^{M^{-1}[\mathcal{S}_1]M\theta_1} \cdots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n}
\end{aligned}$$

Body form of the product of exponentials formula

Screw Axes in the End-Effector Frame



PoE forward kinematics for the 6R open chain

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

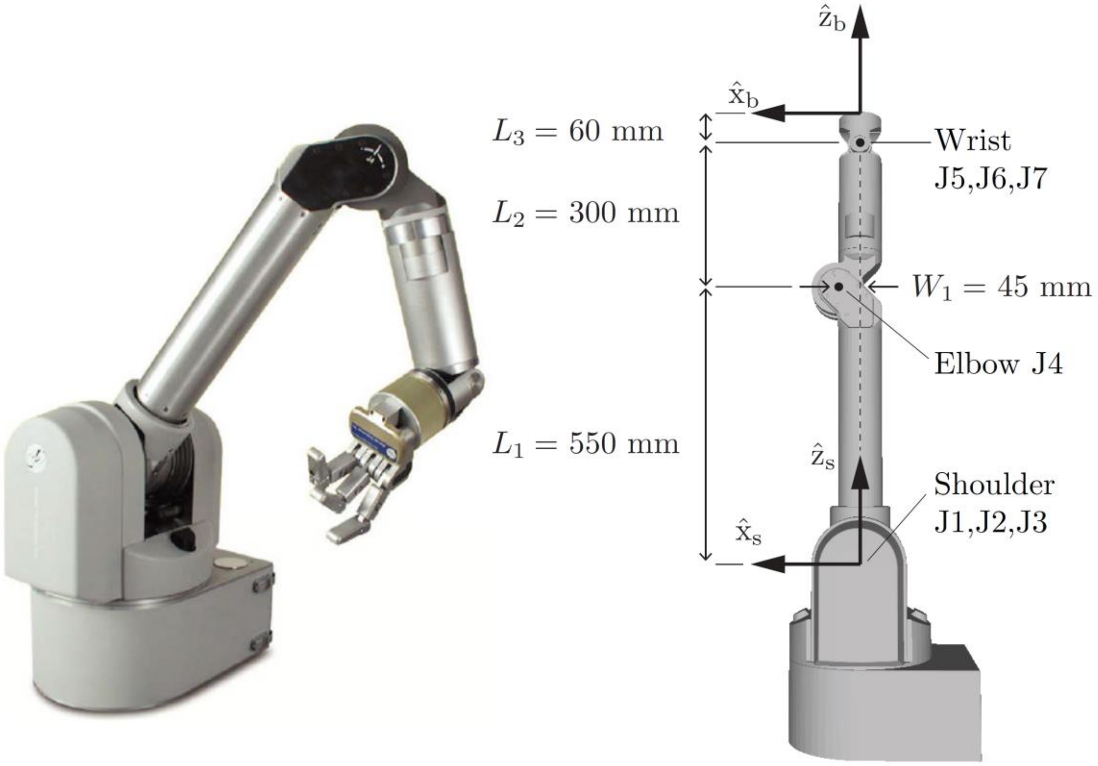
i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,0)$
4	$(-1,0,0)$	$(0,0,L)$
5	$(-1,0,0)$	$(0,0,2L)$
6	$(0,1,0)$	$(0,0,0)$

Space form

i	ω_i	v_i
1	$(0,0,1)$	$(-3L,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,-3L)$
4	$(-1,0,0)$	$(0,0,-2L)$
5	$(-1,0,0)$	$(0,0,-L)$
6	$(0,1,0)$	$(0,0,0)$

Body form

Screw Axes in the End-Effector Frame



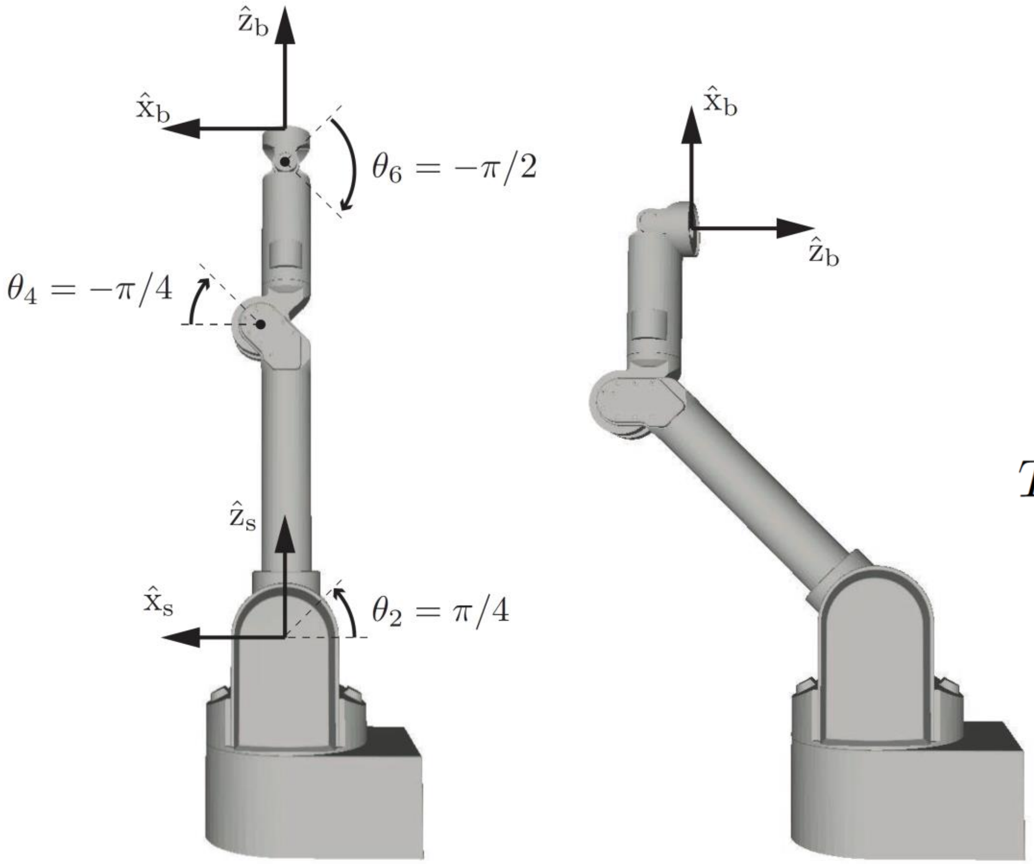
Barrett Technology's WAM 7R robot arm at its zero configuration

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{B}_i = (\omega_i, v_i)$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(L_1 + L_2 + L_3, 0, 0)$
3	$(0,0,1)$	$(0,0,0)$
4	$(0,1,0)$	$(L_2 + L_3, 0, W_1)$
5	$(0,0,1)$	$(0,0,0)$
6	$(0,1,0)$	$(L_3, 0, 0)$
7	$(0,0,1)$	$(0,0,0)$

Screw Axes in the End-Effector Frame



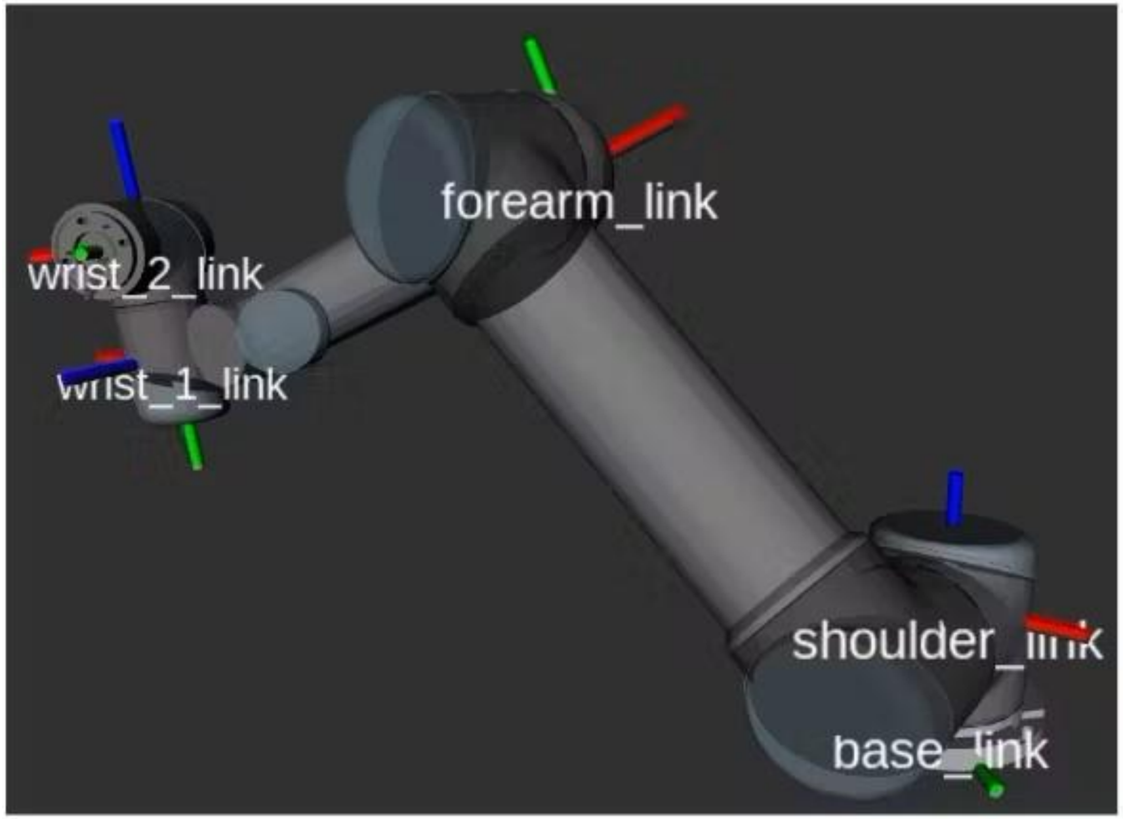
Barrett Technology's WAM 7R robot arm at its zero configuration

$$\theta_2 = 45°,\ \theta_4 = -45°,\ \theta_6 = -90°$$

$$T(\theta) = Me^{[\mathcal{B}_2]\pi/4}e^{-[\mathcal{B}_4]\pi/4}e^{-[\mathcal{B}_6]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.3157 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.6571 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Universal Robot Description Format (URDF)

- URDF is an XML format used by ROS to describe robots



rviz

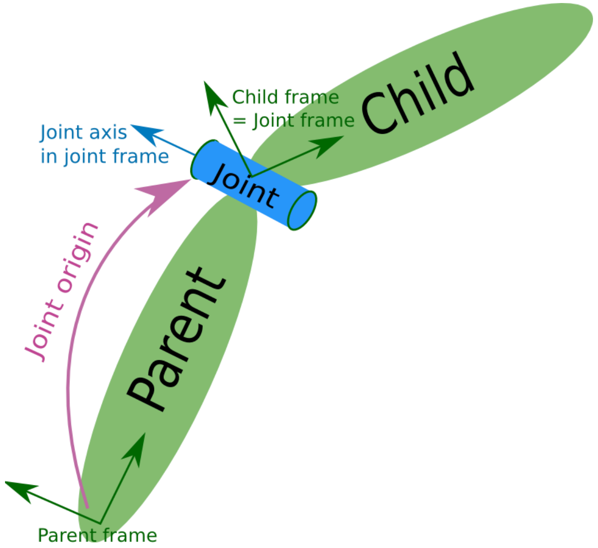
```
<robot name=”ur5”>
  <link name = “base_link”>
     ...
  </link>
  <joint name = “joint1”>
     ...
  </joint>

  ...

  <link name = “wrist_1_link”>
    ...
  </link>
  <joint name = “joint_n”>
    ...
  </joint>
 </robot>
```

URDF

Universal Robot Description Format (URDF)

- Joints

Joint type
- Prismatic
- revolute (including joint limits)
- continuous (revolute without joint limits)
- fixed (virtual joint without motion)




```
<joint name="joint1" type="continuous">
  <parent link="base_link"/>
  <child link="link1"/>
  <origin rpy="0.0 0.0 0.0" xyz="0.0 0.0 0.089159"/>
  <axis xyz="0 0 1"/>
</joint>
```

A unit vector expressed in the child link's frame, in the direction of positive rotation for a revolute joint or positive translation for a prismatic joint.

Origin frame that defines the position and orientation of the child link frame relative to the parent link frame when the joint variable is zero

Origin on the joint axis

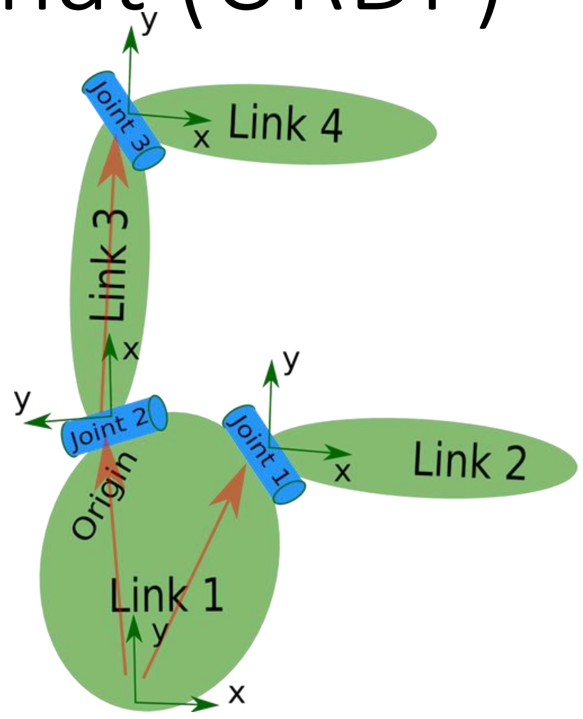
{b} relative to {a}
- Roll: fixed x-axis of {a}
- Pitch: fixed y-axis of {a}
- Yaw: fixed z-axis of {a}

Universal Robot Description Format (URDF)

- Links

an origin frame that defines the position and orientation of a frame at the link's center of mass relative to the link's joint

```
<link name="link1">
  <inertial>
    <mass value="3.7"/>
    <origin rpy="0 0 0" xyz="0.0 0.0 0.0"/>
    <inertia ixx="0.010267495893" ixy="0.0" ixz="0.0"
             iyy="0.010267495893" iyz="0.0" izz="0.00666"/>
  </inertial>
</link>
```

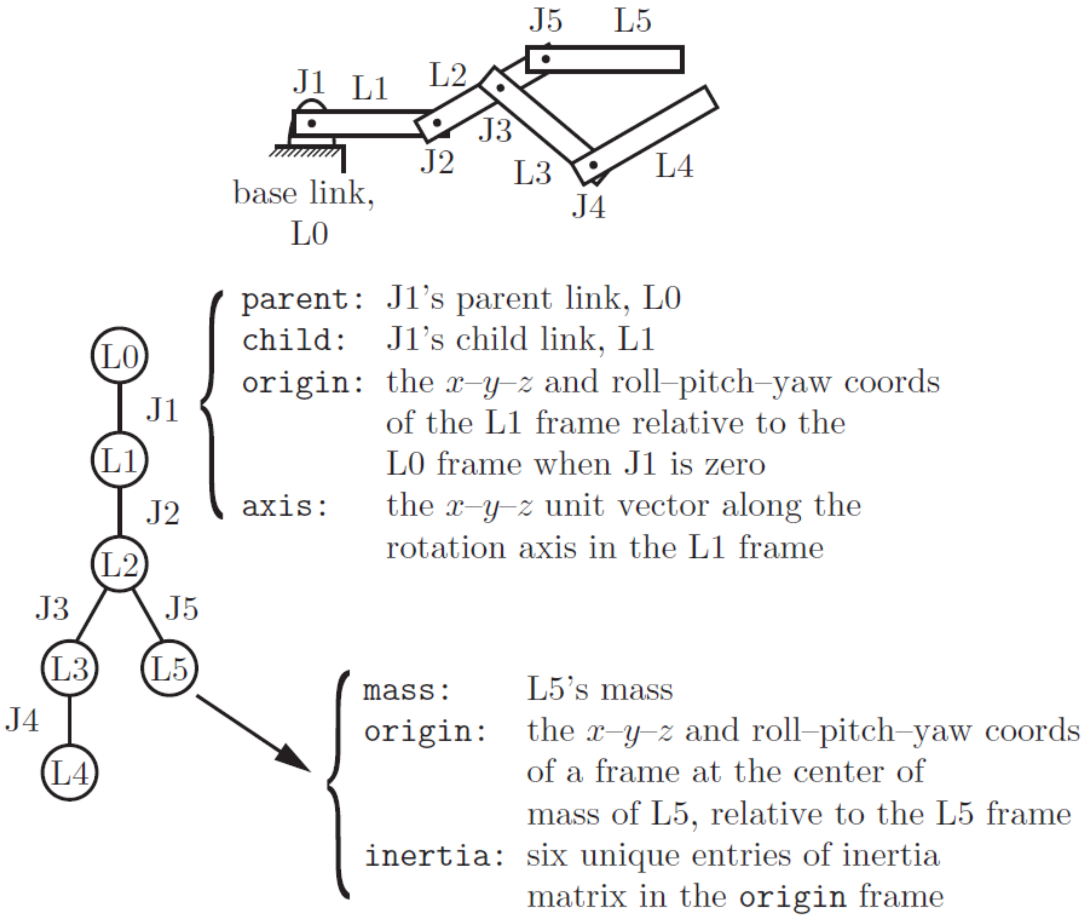


an inertia matrix, relative to the link's center of mass frame

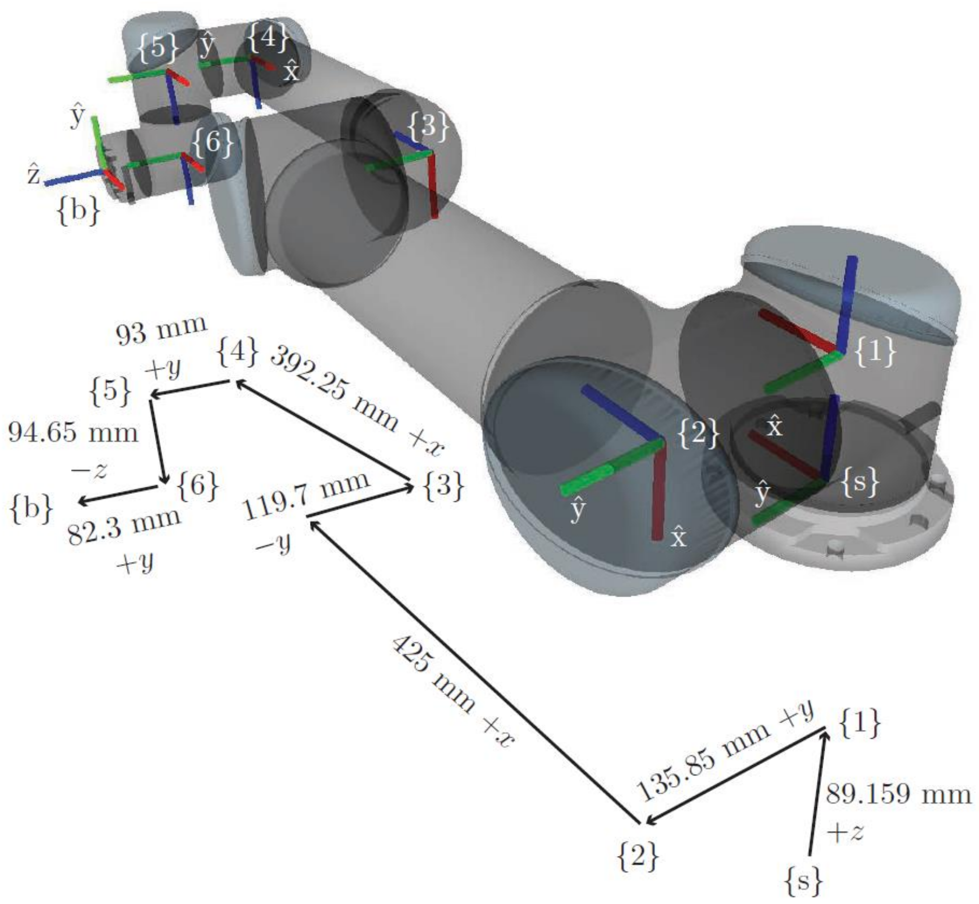
3x3 symmetric positive-definite matrix

Universal Robot Description Format (URDF)

- Can represent any robot with a tree structure



Universal Robot Description Format (URDF)




```
<?xml version="1.0" ?>
<robot name="ur5">

<!-- ********** KINEMATIC PROPERTIES (JOINTS) ********** -->
  <joint name="world_joint" type="fixed">
    <parent link="world"/>
    <child link="base_link"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 0.0 0.0"/>
  </joint>
  <joint name="joint1" type="continuous">
    <parent link="base_link"/>
    <child link="link1"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 0.0 0.089159"/>
    <axis xyz="0 0 1"/>
  </joint>
  <joint name="joint2" type="continuous">
    <parent link="link1"/>
    <child link="link2"/>
    <origin rpy="0.0 1.570796325 0.0" xyz="0.0 0.13585 0.0"/>
    <axis xyz="0 1 0"/>
  </joint>
  <joint name="joint3" type="continuous">
    <parent link="link2"/>
    <child link="link3"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 -0.1197 0.425"/>
    <axis xyz="0 1 0"/>
  </joint>
  <joint name="joint4" type="continuous">
    <parent link="link3"/>
    <child link="link4"/>
    <origin rpy="0.0 1.570796325 0.0" xyz="0.0 0.0 0.39225"/>
    <axis xyz="0 1 0"/>
  </joint>
  <joint name="joint5" type="continuous">
    <parent link="link4"/>
    <child link="link5"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 0.093 0.0"/>
    <axis xyz="0 0 1"/>
  </joint>
  <joint name="joint6" type="continuous">
    <parent link="link5"/>
    <child link="link6"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 0.0 0.09465"/>
    <axis xyz="0 1 0"/>
  </joint>
  <joint name="ee_joint" type="fixed">
    <origin rpy="-1.570796325 0 0" xyz="0 0.0823 0"/>
    <parent link="link6"/>
    <child link="ee_link"/>
  </joint>
```

Summary

- Forward kinematics
- Product of Exponentials Formula
    - Space form
    - Body form
- Universal Robot Description Format (URDF)

Further Reading

- Chapter 4 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.