Forward Kinematics and Product of Exponentials Formula

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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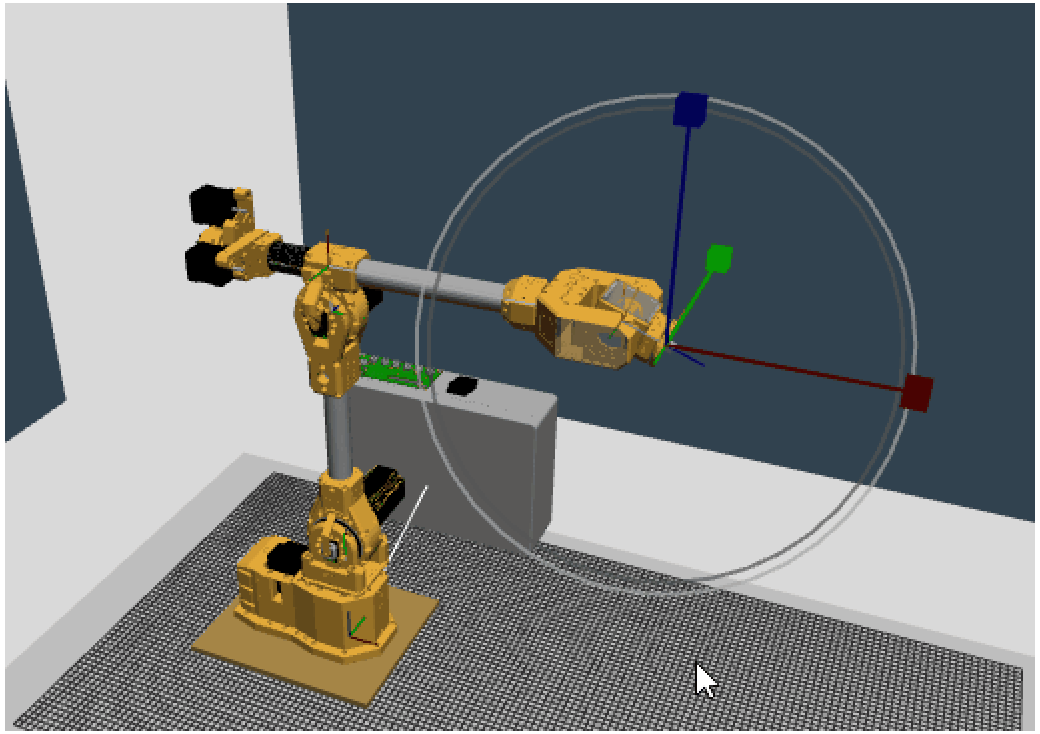
Robot Kinematics

- The relationship between a robot's joint coordinates and its spatial layout



https://www.marginallyclever.com/2020/04/gradient-descent-inverse-kinematics-for-6dof-robot-arms/

Forward Kinematics

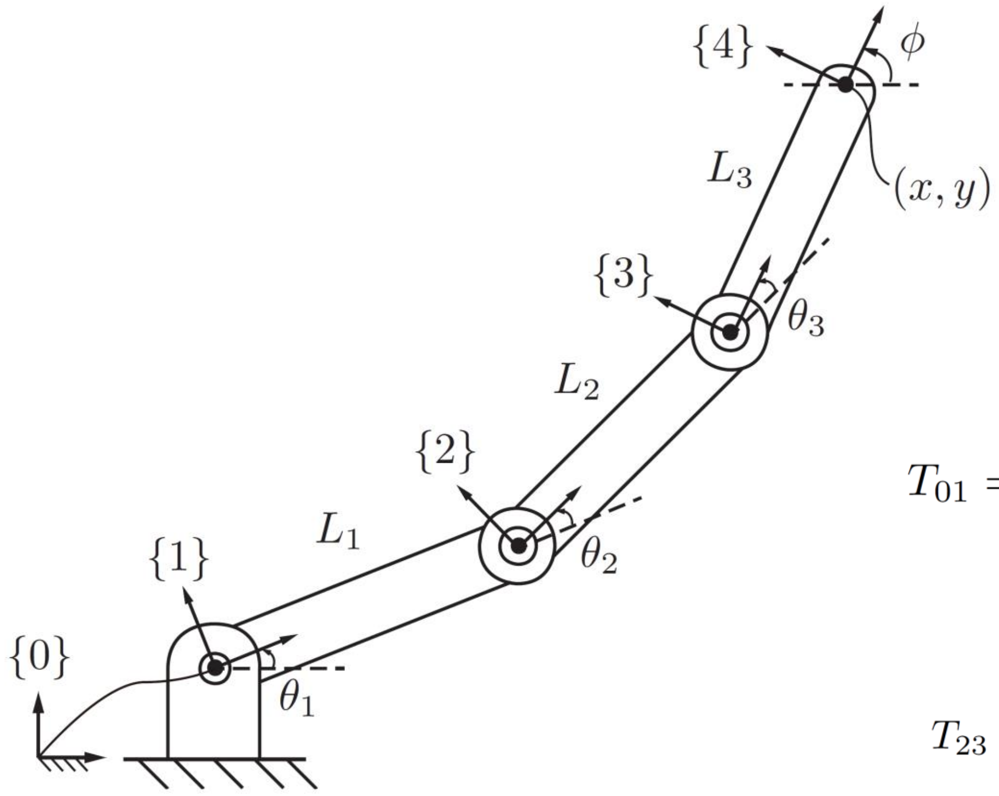
- Forward kinematics of a robot: calculation of the position and orientation of its end-effector from its joint coordinates

End-effector transformation

Joint coordinates θ →

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Recap: Forward Kinematics with D-H Parameters



Forward kinematics of a 3R planar open chain.

- General cases
  - Attaching frames to links
  - Using homogeneous transformations

$$T_{04} = T_{01}T_{12}T_{23}T_{34}$$

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

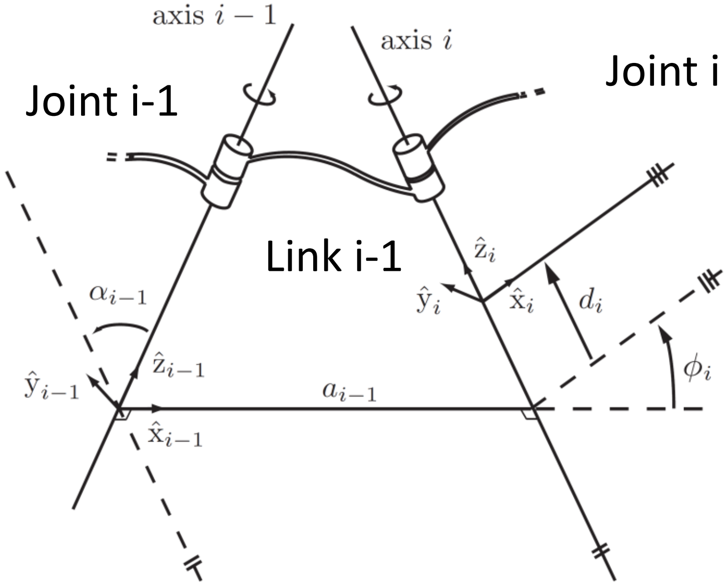
$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$T_{i-1,i}$ Depends only on the joint variable θ_i

Recap: Forward Kinematics with D-H Parameters

$$T_{i-1,i} = \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Trans}(\hat{\mathrm{z}}, d_i)\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i)$$

$$= \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & a_{i-1} \\ \sin\phi_i \cos\alpha_{i-1} & \cos\phi_i \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i \sin\alpha_{i-1} \\ \sin\phi_i \sin\alpha_{i-1} & \cos\phi_i \sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i \cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$T_{0n}(\theta_1, \ldots, \theta_n) = T_{01}(\theta_1)T_{12}(\theta_2)\cdots T_{n-1,n}(\theta_n)$$

$$T_{i,i-1} \in SE(3)$$

Forward Kinematics

- Method 1: uses homogeneous transformations
  - Need to define the coordinates of frames
  - Denavit-Hartenberg Parameters

- Method 2: uses screw-axis representations of transformations
  - No need to define frame references

Screw-Axis Representations

- Screw axis: motion of a screw

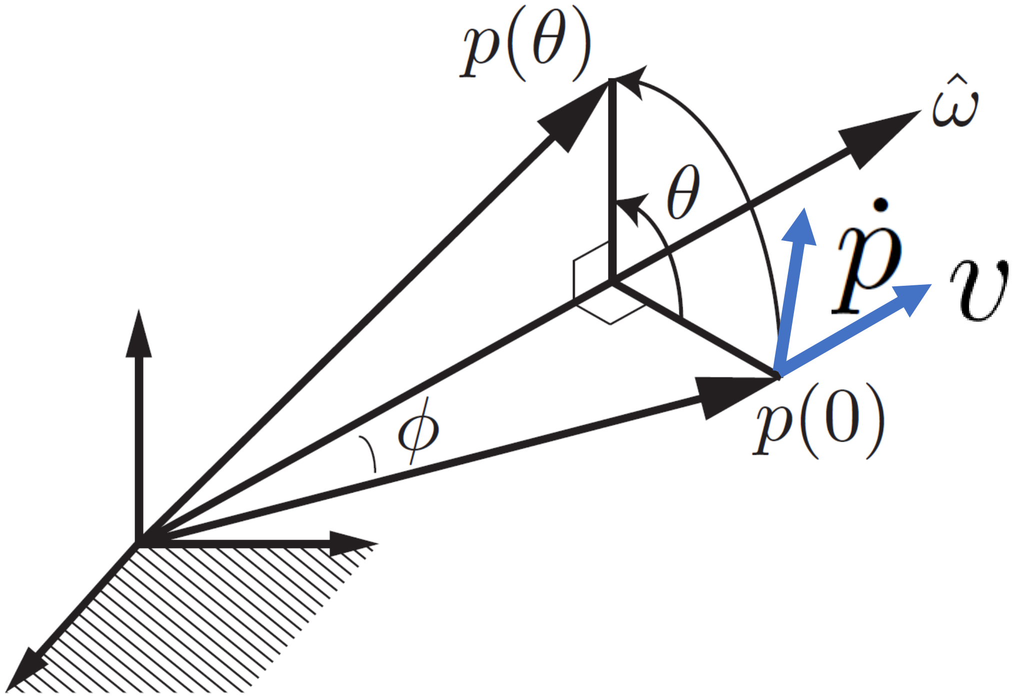
https://mecharithm.com/learning/lesson/screw-motion-and-exponential-coordinates-of-robot-motions-11

- **Chasles-Mozzi theorem**: every rigid-body displacement can be expressed as displacement along a fixed screw axis S in space

Exponential Coordinates of Rigid-Body Motions

- p(0) is rotated to $p(\theta)$
  - At a constant rate of 1 rad/s

- p(t): path traced by the tip of vector



Velocity

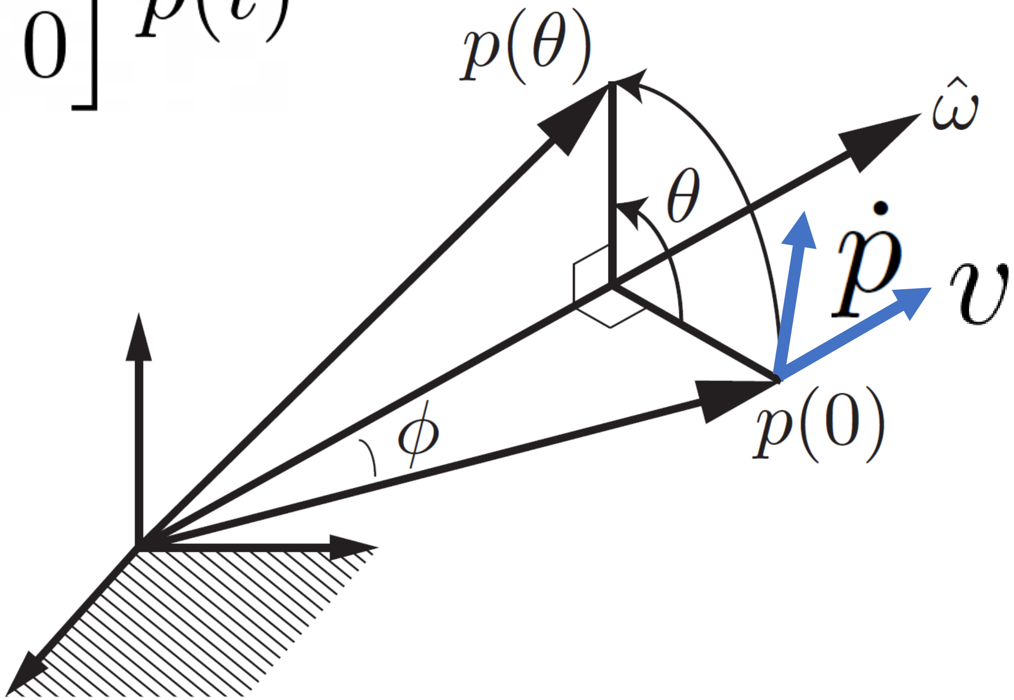
$$\dot{p}(t) = v + \hat{\omega} \times p(t)$$

Exponential Coordinates of Rigid-Body Motions

$$\dot{p}(t) = v + \hat{\omega} \times p(t) \qquad \dot{\widetilde{p}}(t) = \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix} \widetilde{p}(t)$$

$$[\mathcal{S}] = \begin{bmatrix} [\hat{\omega}] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

$$\dot{\widetilde{p}}(t) = [\mathcal{S}]\widetilde{p}(t)$$



Solution $$\widetilde{p}(t) = e^{[\mathcal{S}]t}\widetilde{p}(0)$$

Transformation $$T(t) = e^{[\mathcal{S}]t}$$

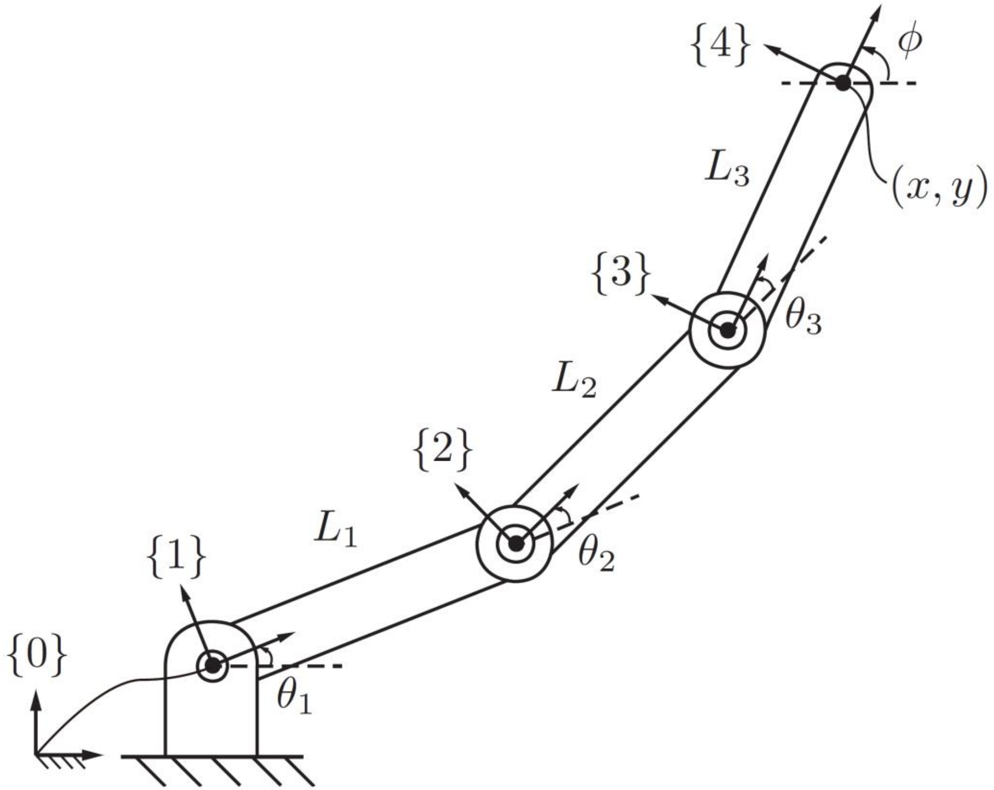
Exponential Coordinates of Rigid-Body Motions

$$T(t) = e^{[\mathcal{S}]t}$$

If $\|\omega\| = 1$ $\quad e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)\, v \\ 0 & 1 \end{bmatrix}$

If $\omega = 0$ and $\|v\| = 1$ $\quad e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$

Forward Kinematics

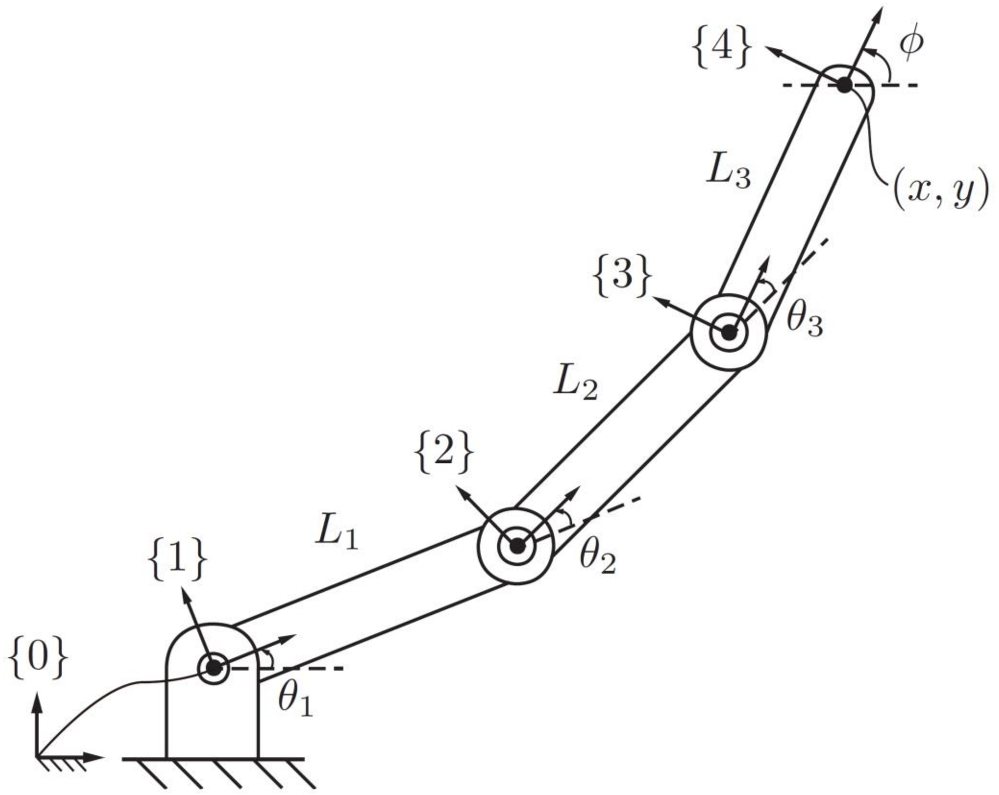


Forward kinematics of a 3R planar open chain.

- A different approach
- Define M to the position and orientation of frame {4} when all the joint angles are zeros (“home” or “zero” position of the robot)

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics



Forward kinematics of a 3R planar open chain.

- Consider each revolute joint as a zero-pitch screw-axis expressed in the {0} frame (fixed frame)

For joint 3

Spatial twist

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} \qquad \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

v_3 Linear velocity of the origin of {0} in the {0} frame when joint 3 rotates

$$v_3 = -\omega_3 \times q_3$$

$$q_3 = (L_1 + L_2, 0, 0)$$

$$v_3 = \begin{bmatrix} 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

Cross Product

- Matrix notation

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\ &= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}, \end{aligned}$$

https://en.wikipedia.org/wiki/Cross_product

Forward Kinematics

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = \theta_2 = 0)$$

$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = 0)$$

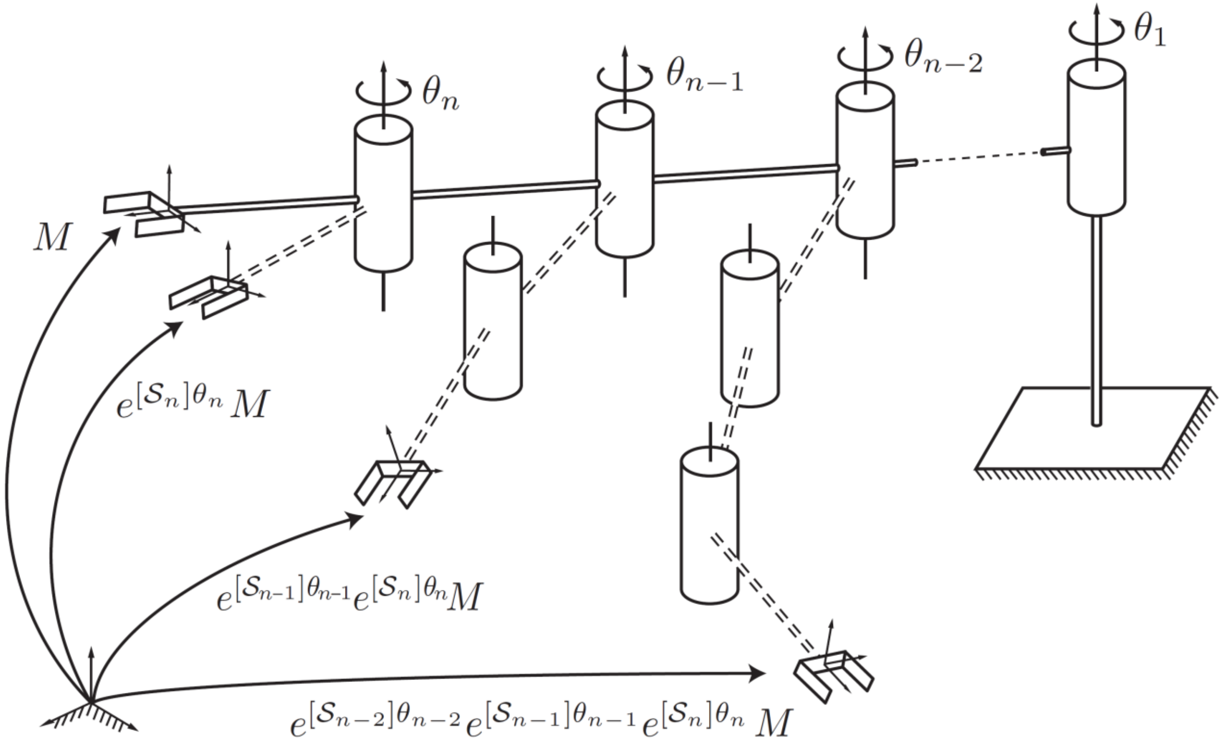
$$[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

a product of matrix exponentials
(does not use any frame references, only {0} and M)

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Product of Exponentials Formula



- Each link apply a screw motion to all the outward links
- Base frame {s}
- End-effector frame {b}

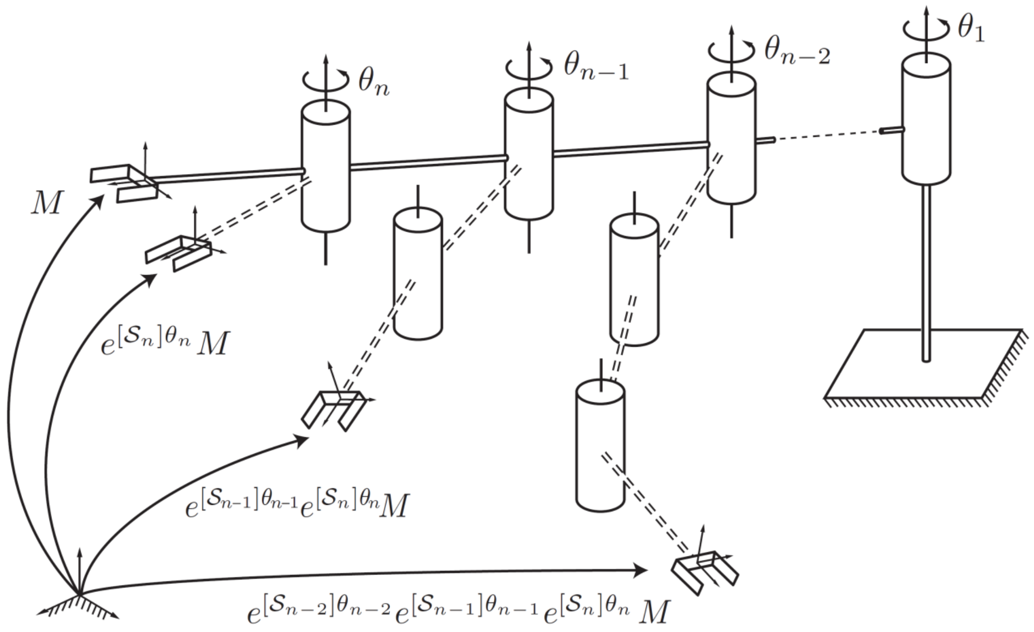
$$M \in SE(3)$$

{b} in {s} when all the joint values are zeros

$$T = e^{[\mathcal{S}_n]\theta_n} M$$

{b} in {s} when joint n with value θ_n

Product of Exponentials Formula

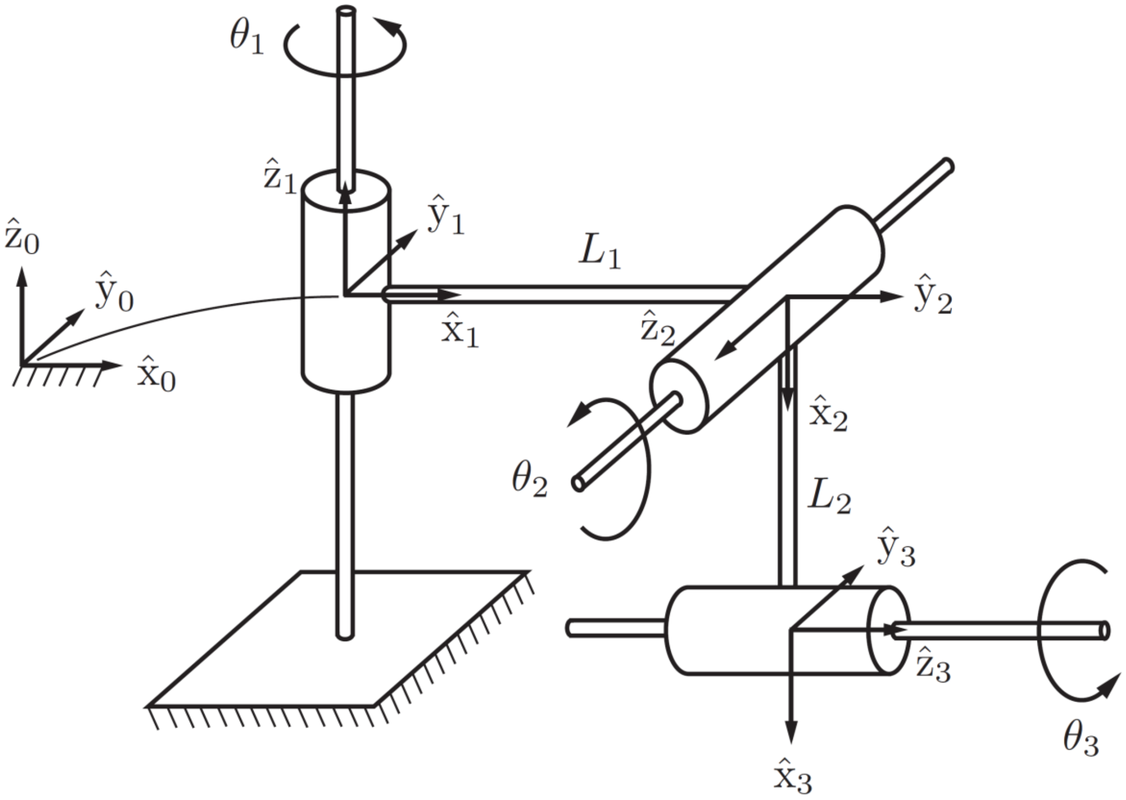


$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Joint values $(\theta_1, \ldots, \theta_n)$

- Space form of the product of exponentials formula
- Unlike D-H representation, no link reference frames need to be defined

Product of Exponentials Formula



A 3R spatial open chain

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{S}_1 = (\omega_1, v_1) \quad \omega_1 = (0,0,1) \quad v_1 = (0,0,0)$$

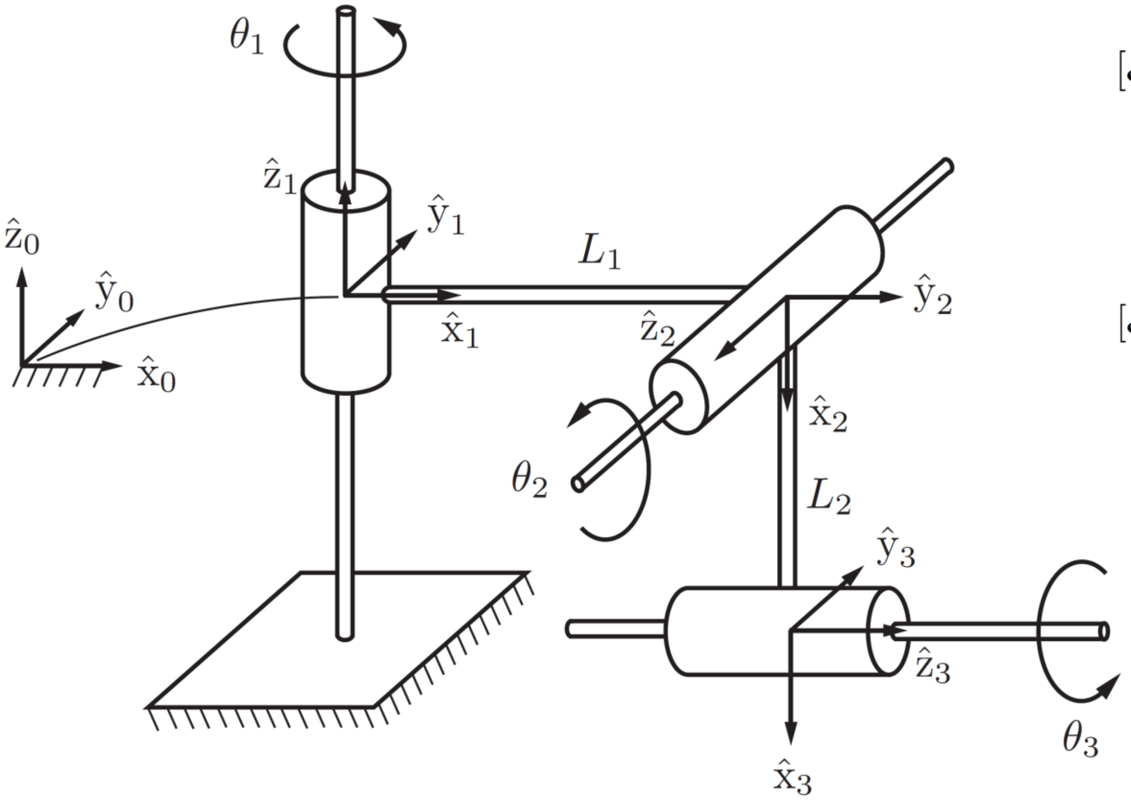
$$\omega_2 = (0,-1,0) \quad q_2 = (L_1, 0, 0)$$

$$v_2 = -\omega_2 \times q_2 = (0, 0, -L_1)$$

$$\omega_3 = (1,0,0) \quad q_3 = (0,0,-L_2)$$

$$v_3 = -\omega_3 \times q_3 = (0, -L_2, 0)$$

Product of Exponentials Formula



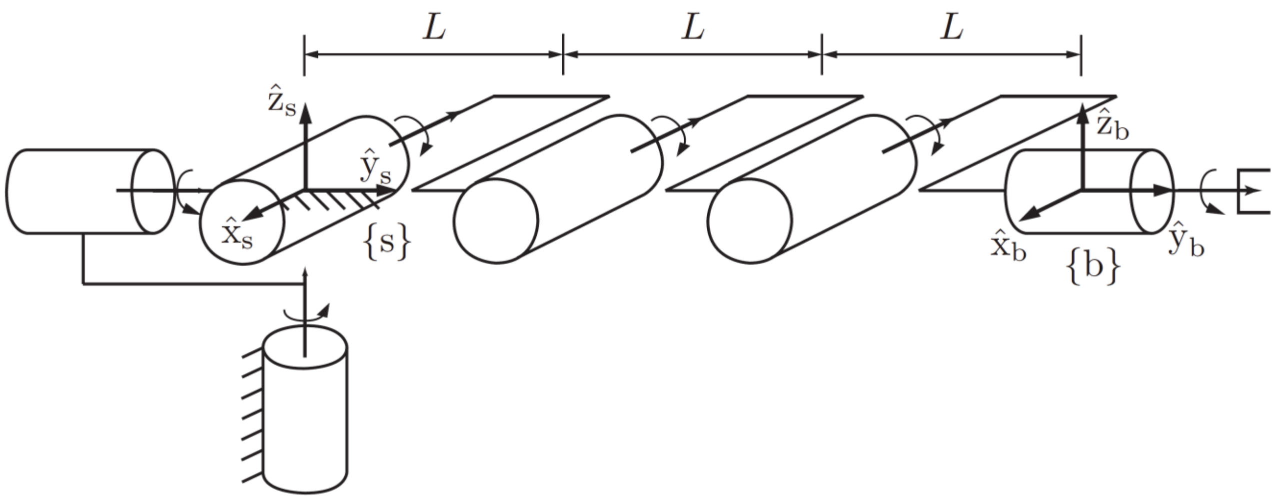
A 3R spatial open chain

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, -1, 0)$	$(0, 0, -L_1)$
3	$(1, 0, 0)$	$(0, L_2, 0)$

Product of Exponentials Formula



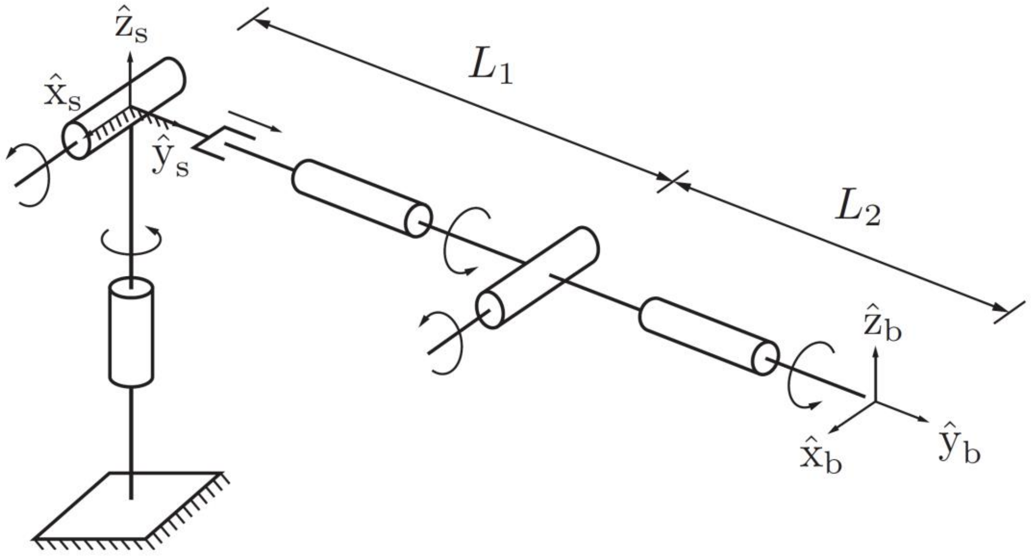
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, 1, 0)$	$(0, 0, 0)$
3	$(-1, 0, 0)$	$(0, 0, 0)$
4	$(-1, 0, 0)$	$(0, 0, L)$
5	$(-1, 0, 0)$	$(0, 0, 2L)$
6	$(0, 1, 0)$	$(0, 0, 0)$

PoE forward kinematics for the 6R open chain

First three joints are at the same location

Product of Exponentials Formula

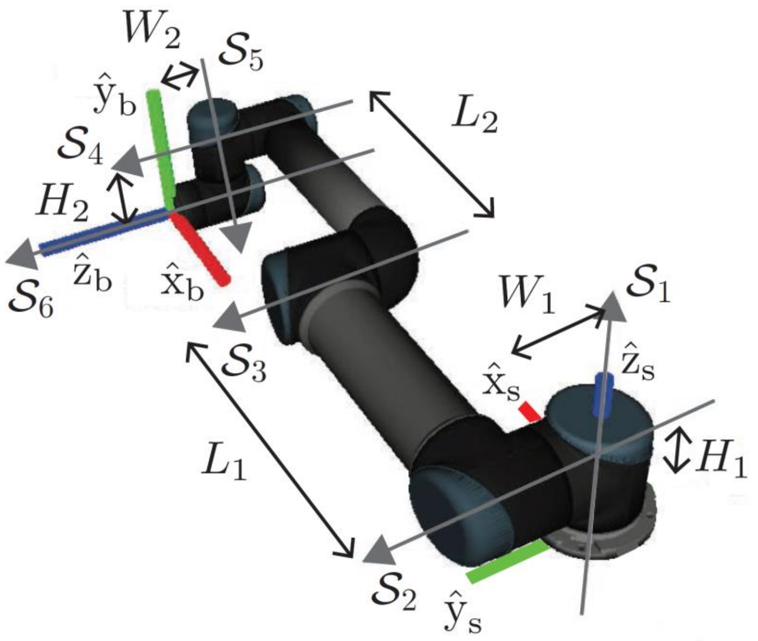


The RRPRRR spatial open chain

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(1, 0, 0)$	$(0, 0, 0)$
3	$(0, 0, 0)$	$(0, 1, 0)$
4	$(0, 1, 0)$	$(0, 0, 0)$
5	$(1, 0, 0)$	$(0, 0, -L_1)$
6	$(0, 1, 0)$	$(0, 0, 0)$

Product of Exponentials Formula



Universal Robots' UR5 6R robot arm

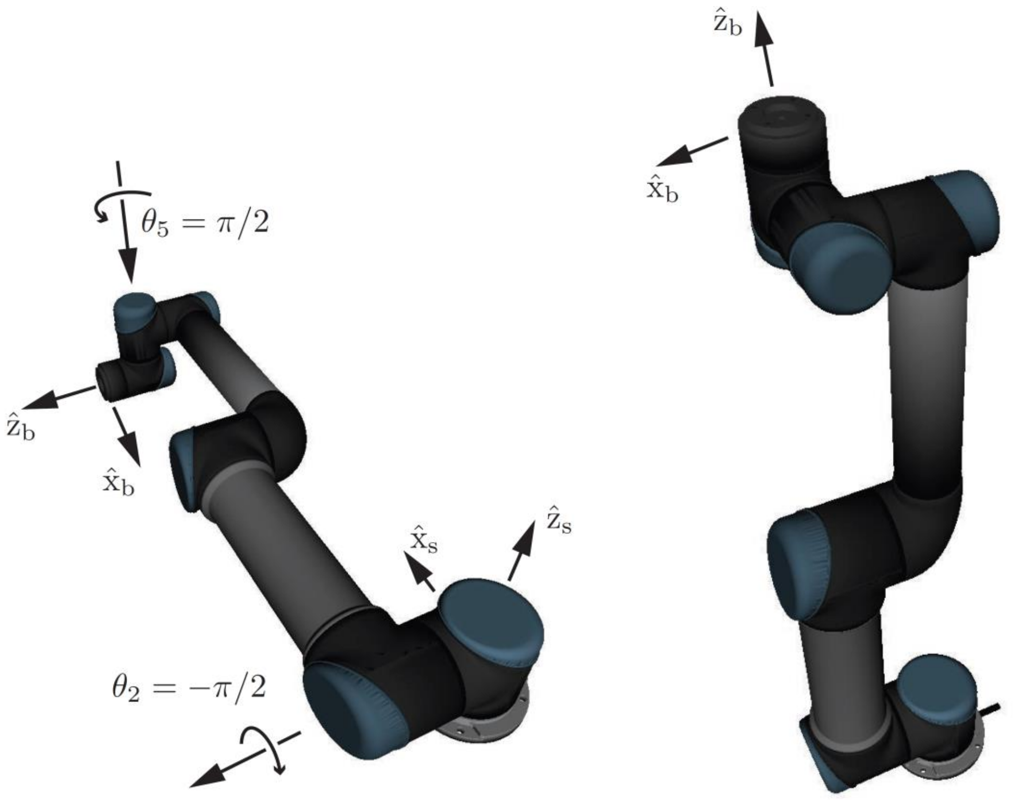
$$W_1 = 109 \text{ mm},\ W_2 = 82 \text{ mm},\ L_1 = 425 \text{ mm}$$

$$L_2 = 392 \text{ mm},\ H_1 = 89 \text{ mm} \quad H_2 = 95 \text{ mm}$$

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1,0,0)$
3	$(0,1,0)$	$(-H_1,0,L_1)$
4	$(0,1,0)$	$(-H_1,0,L_1+L_2)$
5	$(0,0,-1)$	$(-W_1,L_1+L_2,0)$
6	$(0,1,0)$	$(H_2-H_1,0,L_1+L_2)$

Product of Exponentials Formula



Universal Robots' UR5 6R robot arm

$W_1 = 109$ mm, $W_2 = 82$ mm, $L_1 = 425$ mm

$L_2 = 392$ mm, $H_1 = 89$ mm $H_2 = 95$ mm

$$\theta_2 = -\pi/2 \text{ and } \theta_5 = \pi/2.$$

$$\begin{aligned} T(\theta) &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M \\ &= I e^{-[\mathcal{S}_2]\pi/2} I^2 e^{[\mathcal{S}_5]\pi/2} I M \\ &= e^{-[\mathcal{S}_2]\pi/2} e^{[\mathcal{S}_5]\pi/2} M \end{aligned}$$

$$e^{-[\mathcal{S}_2]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.089 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.089 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{[\mathcal{S}_5]\pi/2} = \begin{bmatrix} 0 & 1 & 0 & 0.708 \\ -1 & 0 & 0 & 0.926 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T(\theta) = e^{-[\mathcal{S}_2]\pi/2} e^{[\mathcal{S}_5]\pi/2} M = \begin{bmatrix} 0 & -1 & 0 & 0.095 \\ 1 & 0 & 0 & 0.109 \\ 0 & 0 & 1 & 0.988 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary

- Forward kinematics
- Product of Exponentials Formula
  - Space form

Further Reading

- Chapter 4 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.