Twists and Screw Axes

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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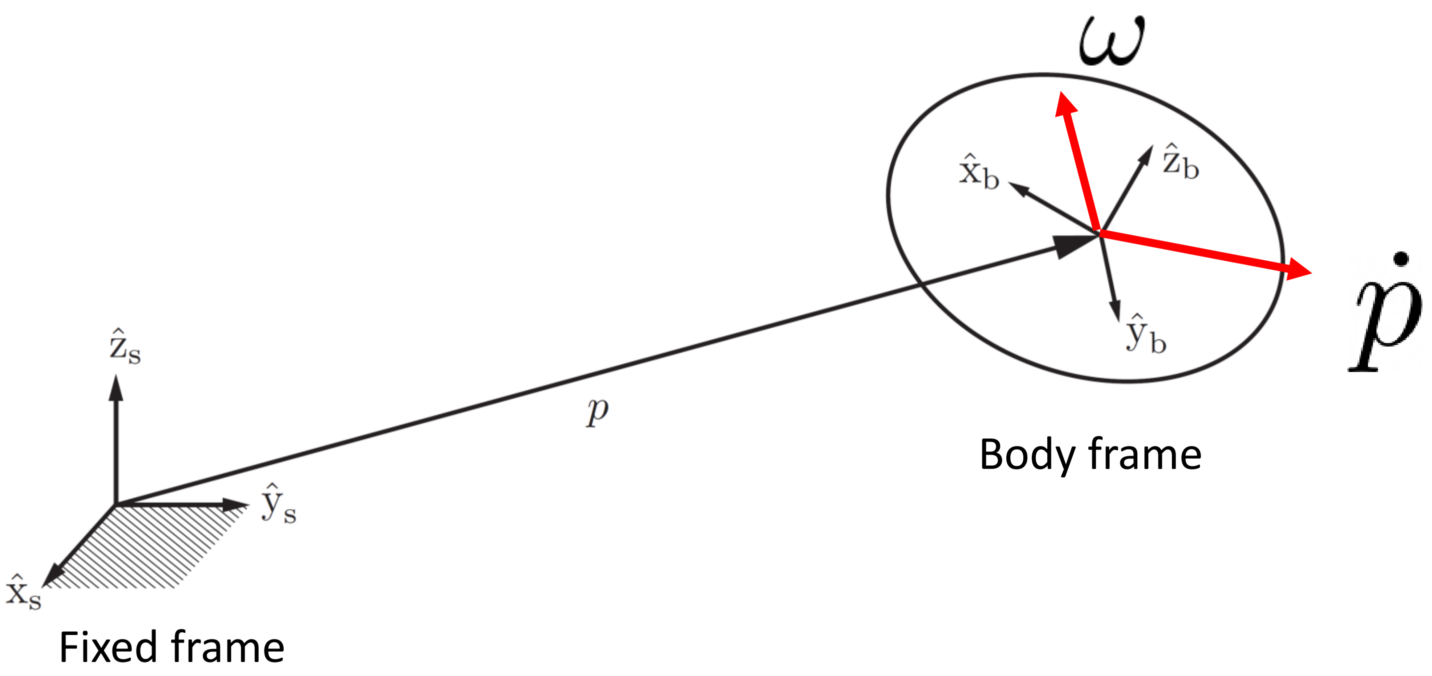
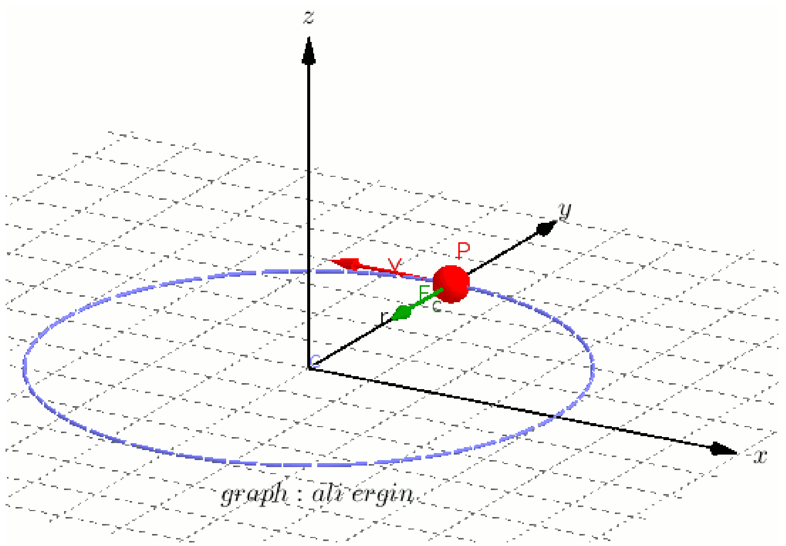
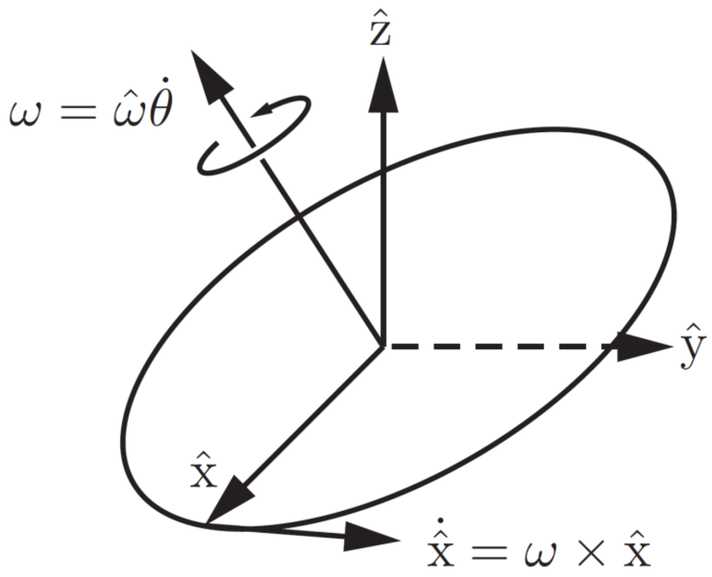
Angular Velocity and Linear Velocity





Recall Angular Velocities



- Angular velocity $\omega = \hat{\omega}\dot{\theta}$
- Compute time derivates of these axes caused by rotation $\dot{\hat{x}}$ (tangential velocity)

$$\dot{\hat{\mathbf{x}}} = \omega \times \hat{\mathbf{x}}$$
$$\dot{\hat{\mathbf{y}}} = \omega \times \hat{\mathbf{y}}$$
$$\dot{\hat{\mathbf{z}}} = \omega \times \hat{\mathbf{z}}$$

$$\dot{R}R^{-1} = [\omega_s]$$
$$R^{-1}\dot{R} = [\omega_b]$$

$$\dot{R} = \omega_s \times R$$

Body Twist

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \qquad [\omega_b] = R^T \dot{R}$$

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \qquad v_b = R^T \dot{p}$$

Body twist $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$

Spatial Twist

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \qquad [\omega_s] = \dot{R}R^{\mathrm{T}}$$

$$\dot{T}T^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \qquad v_s = \dot{p} - \dot{R}R^{\mathrm{T}}p$$

Spatial twist $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6$

Relationship between Body Twist and Spatial Twist

$$
\begin{aligned}
[\mathcal{V}_b] &= T^{-1}\dot{T} \\
&= T^{-1}[\mathcal{V}_s]T
\end{aligned}
$$

$$[\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}$$

$$[\mathcal{V}_s] = \begin{bmatrix} R[\omega_b]R^{\mathrm{T}} & -R[\omega_b]R^{\mathrm{T}}p + Rv_b \\ 0 & 0 \end{bmatrix}$$

$$R[\omega]R^{\mathrm{T}} = [R\omega]$$

$$[\omega]p = -[p]\omega$$

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$6 \times 6$$

Adjoint Representations

- The adjoint representation of $T = (R, p) \in SE(3)$

$$[\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}$$

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Twists

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$$

$$R[\omega]R^{\mathrm{T}} = [R\omega]$$

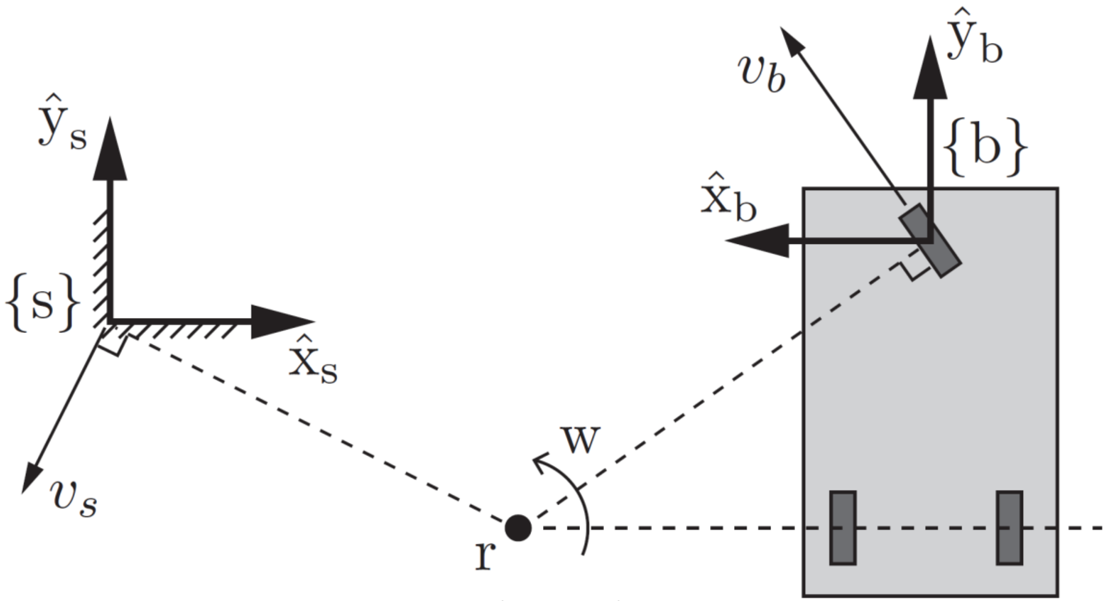
$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s$$

In general

$$\mathcal{V}_c = [\mathrm{Ad}_{T_{cd}}]\mathcal{V}_d, \qquad \mathcal{V}_d = [\mathrm{Ad}_{T_{dc}}]\mathcal{V}_c$$

Twists Example



- Pure Angular velocity $\mathrm{w} = 2 \text{ rad/s}$

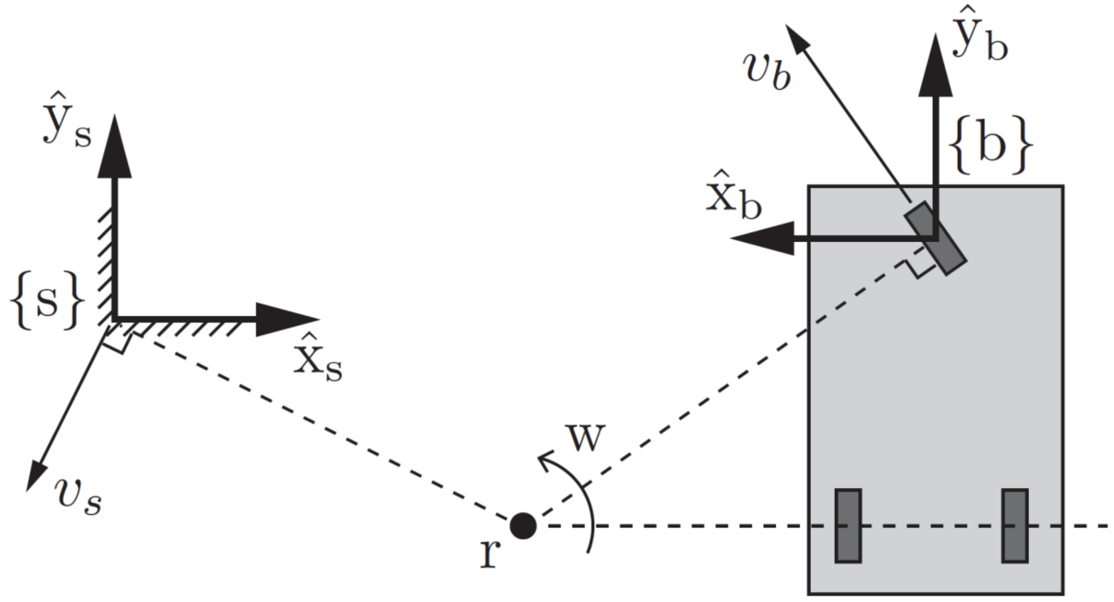
$$r_s = (2, -1, 0) \quad r_b = (2, -1.4, 0)$$

$$\omega_s = (0, 0, 2) \quad \omega_b = (0, 0, -2)$$

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What are the linear velocities? $v_s \quad v_b$

Twists Example



Top view of a car

- Pure Angular velocity $\mathrm{w} = 2 \text{ rad/s}$

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Linear velocity of the car

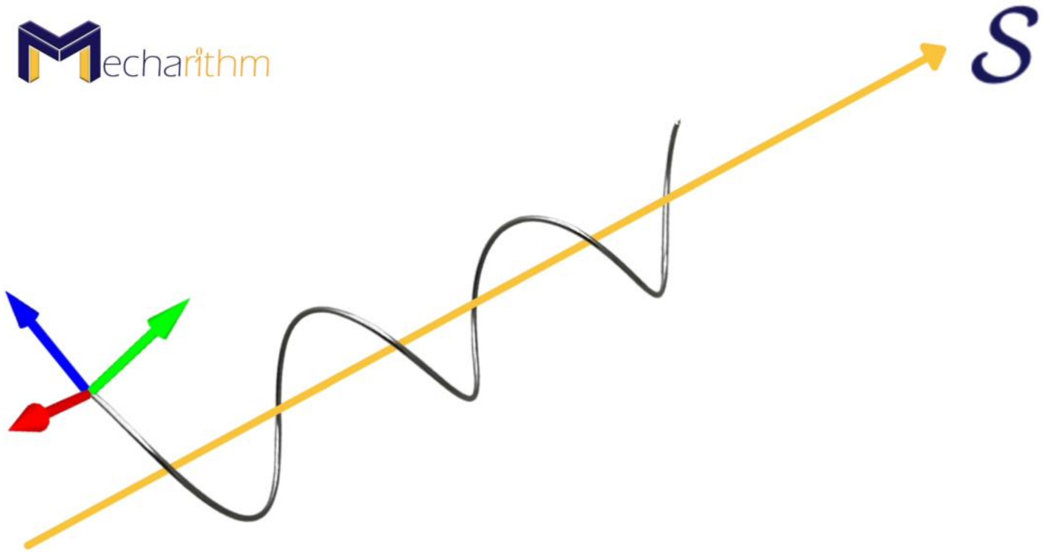
$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2, -4, 0),$$

$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8, 4, 0),$$

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \quad \mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix}$$

The Screw Interpretation of a Twist

- Screw axis: motion of a screw
  - Rotating about the axis while translating along the axis



https://mecharithm.com/learning/lesson/screw-motion-and-exponential-coordinates-of-robot-motions-11

The Screw Interpretation of a Twist

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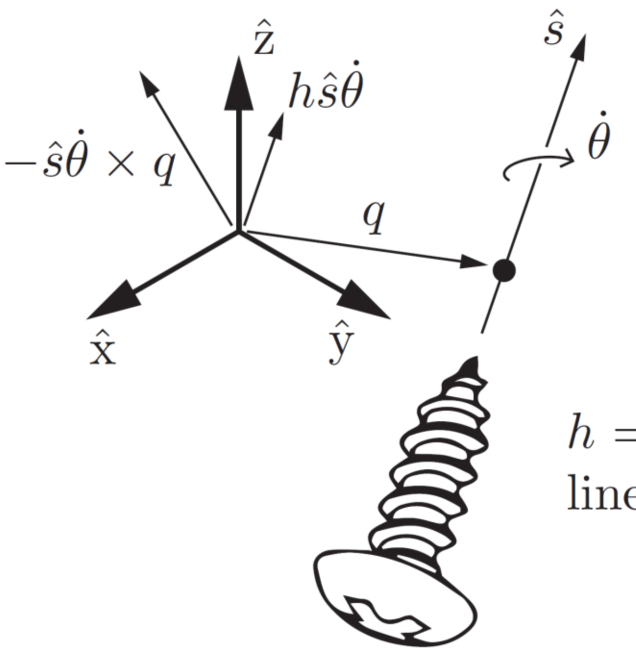

$h = \text{pitch} = \text{linear speed/angular speed}$

Screw axis $\mathcal{S}$ is the collection $\{q, \hat{s}, h\}$

$q \in \mathbb{R}^3$ is a point on the axis (any point is fine)

Twist about S with angular velocity $\dot{\theta}$

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}$$

The Screw Interpretation of a Twist

- For any twist $\mathcal{V} = (\omega, v)$ $\omega \neq 0$
- These exists $\{q, \hat{s}, h\}$ $\dot{\theta}$

$$\hat{s} = \omega / \|\omega\| \qquad \dot{\theta} = \|\omega\| \qquad h = \hat{\omega}^{\mathrm{T}} v / \dot{\theta}$$

portion of v parallel to the screw axis

$-\hat{s}\dot{\theta} \times q$ provides the portion of v orthogonal to the screw axis
(choose q based on this term)

If $\omega = 0$ $\quad \hat{s} = v / \|v\|$ $\quad h = \text{pitch} = \text{linear speed/angular speed}$ infinity

$\dot{\theta}$ is interpreted as the linear velocity $\|v\|$ along $\hat{s}$

Summary

- Twists

- Screw Axes

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017
- Basics of Classical Lie Groups: The Exponential Map, Lie Groups, and Lie Algebras https://www.cis.upenn.edu/~cis6100/geombchap14.pdf
- Exponential Coordinate of Rigid Body Configuration. Prof. Wei Zhang, Southern University of Science and Technology, Shenzhen, China http://www.wzhanglab.site/wp-content/uploads/2021/06/LN4_ExpCoordinate-a-print.pdf