Homogenous Transformations and Twists

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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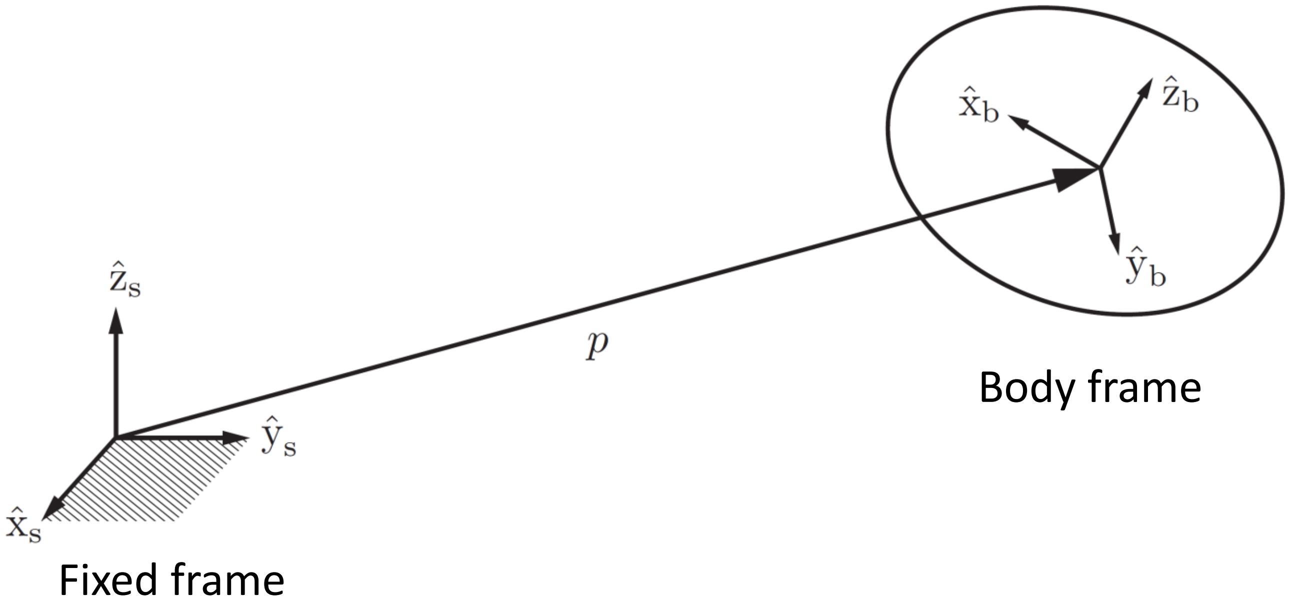
Homogenous Transformation Matrices

- Consider body frame {b} in a fixed frame {s}
  - 3D rotation $R \in SO(3)$
  - 3D position $p \in \mathbb{R}^3$



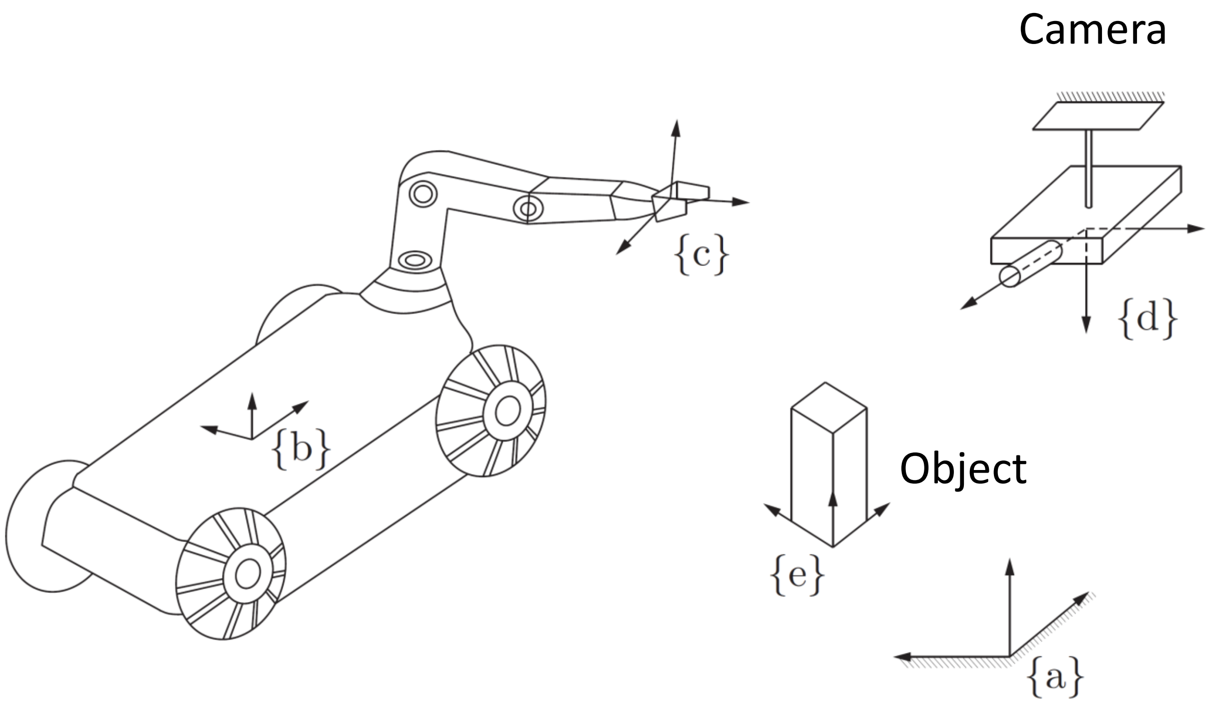
Homogenous Transformation Matrices

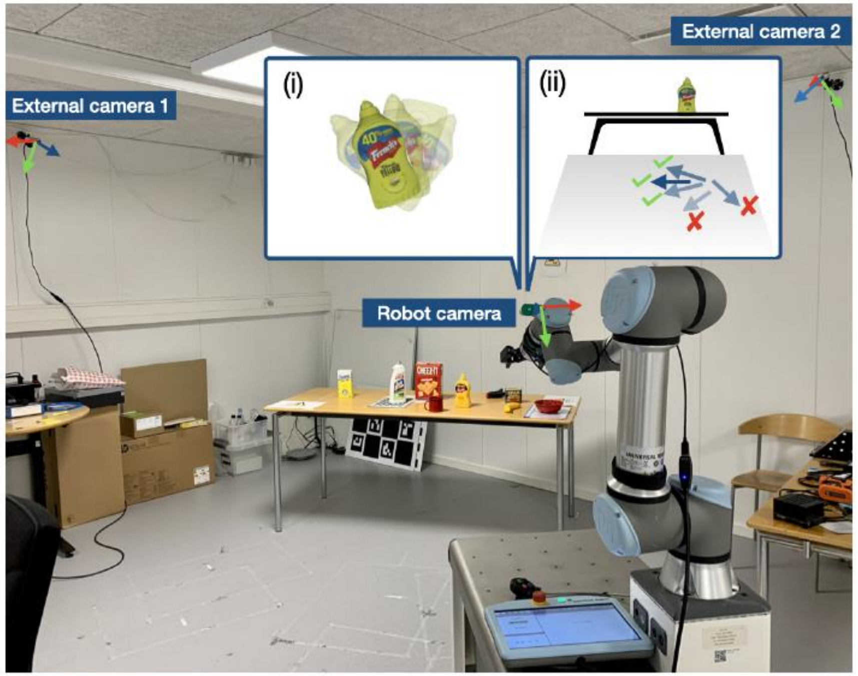
- Consider body frame {b} in a fixed frame {s}
  - 3D rotation $R \in SO(3)$
  - 3D position $p \in \mathbb{R}^3$

- Special Euclidean group SE(3) or homogenous transformation matrices

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices in Robotics



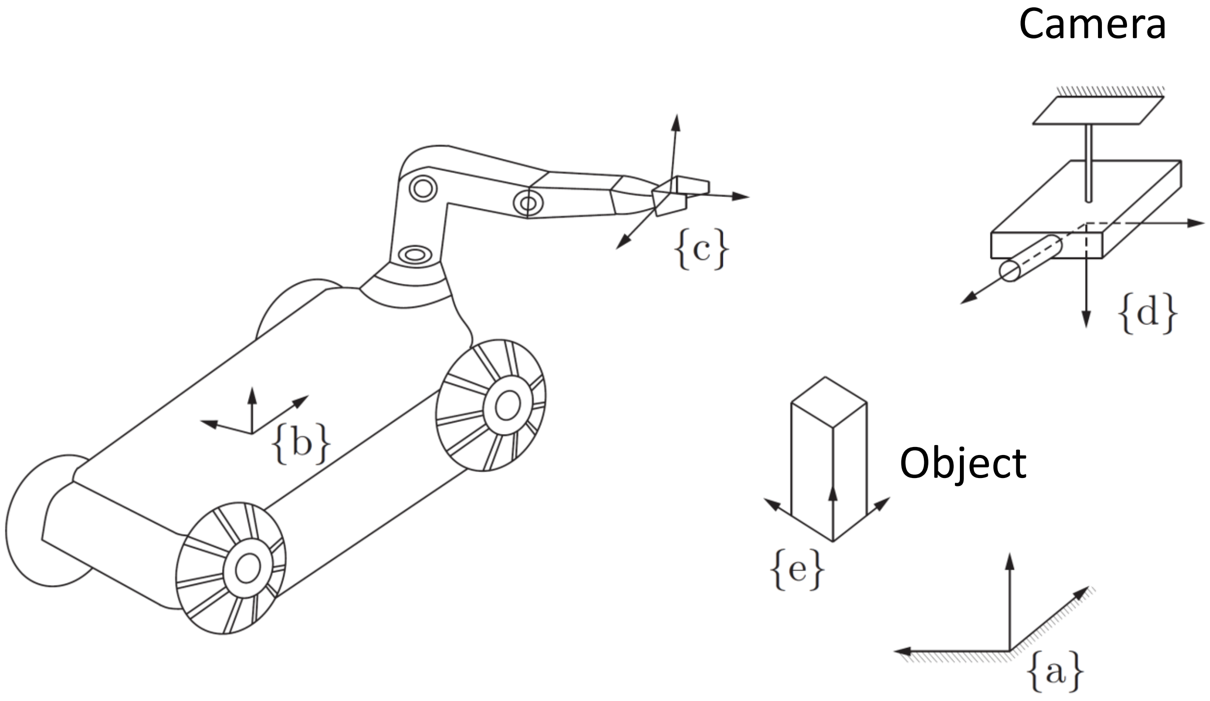


Multi-view object pose distribution tracking for pre-grasp planning on mobile robots. Naik et al., ICRA, 2022.

- How to move the robot arm to pick up the object?
- What information we need to know?

T_{ce} Pose of object in gripper

Transformation Matrices



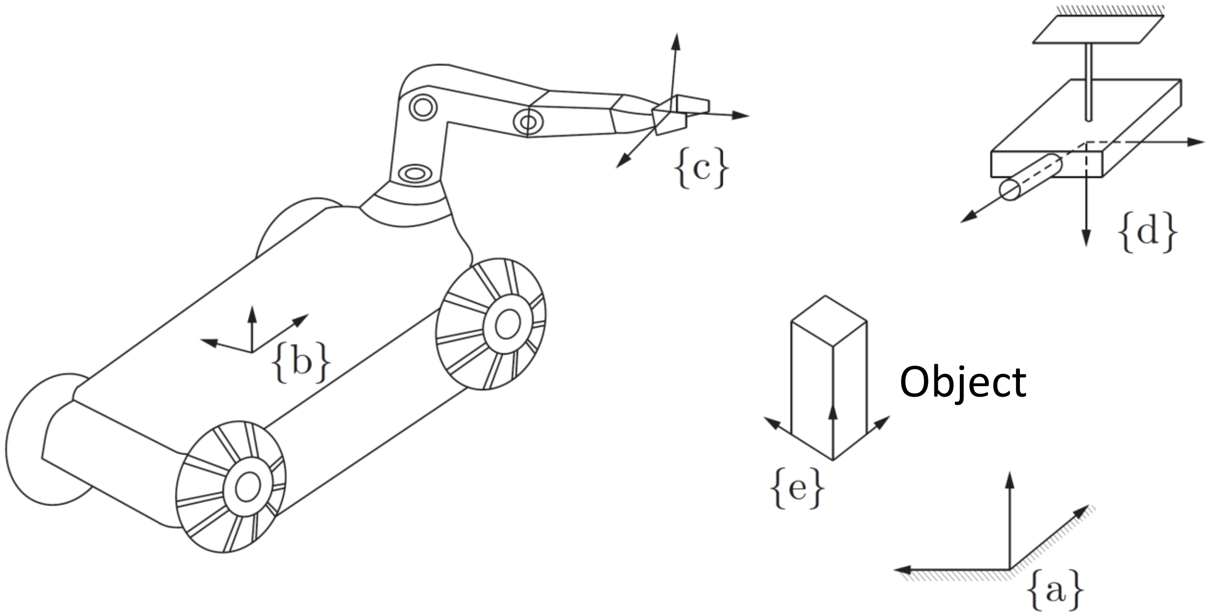
- Robot tracking in camera

https://research.nvidia.com/publication/2020-05_camera-robot-pose-estimation-single-image

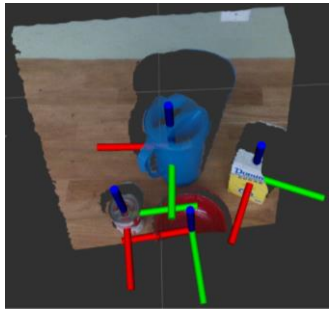
Robot in camera

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



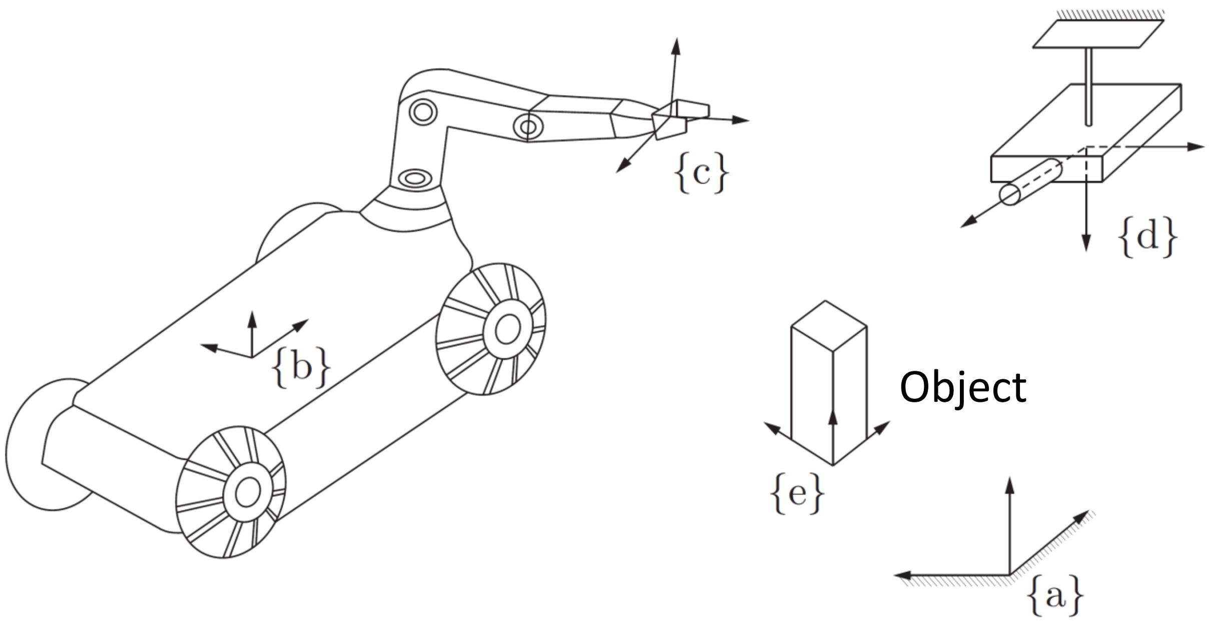
- Object pose estimation from camera



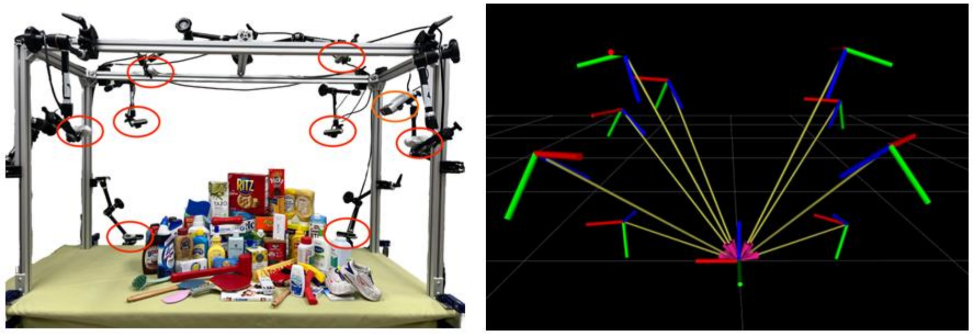
Object in camera

$$T_{de} = \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



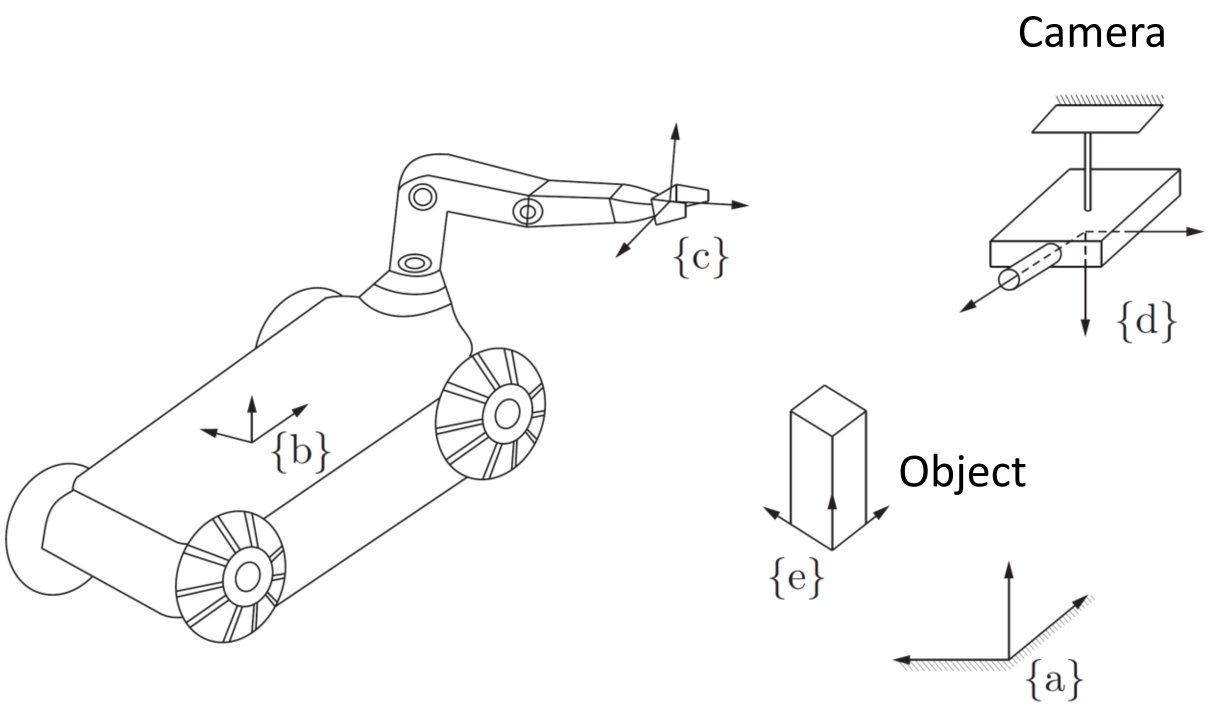
- Camera calibration



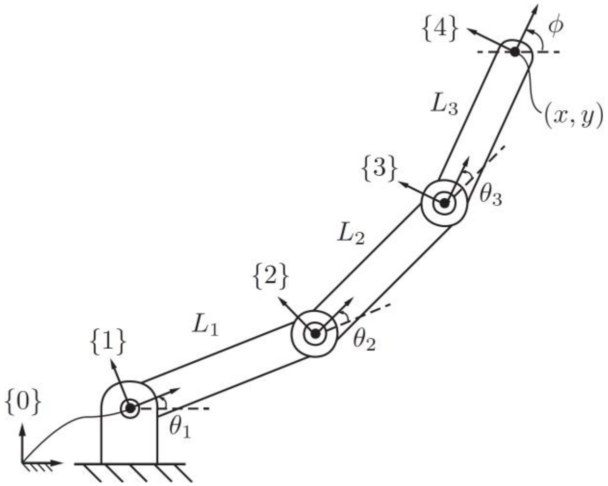
Camera in fixed frame

$$T_{ad} = \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



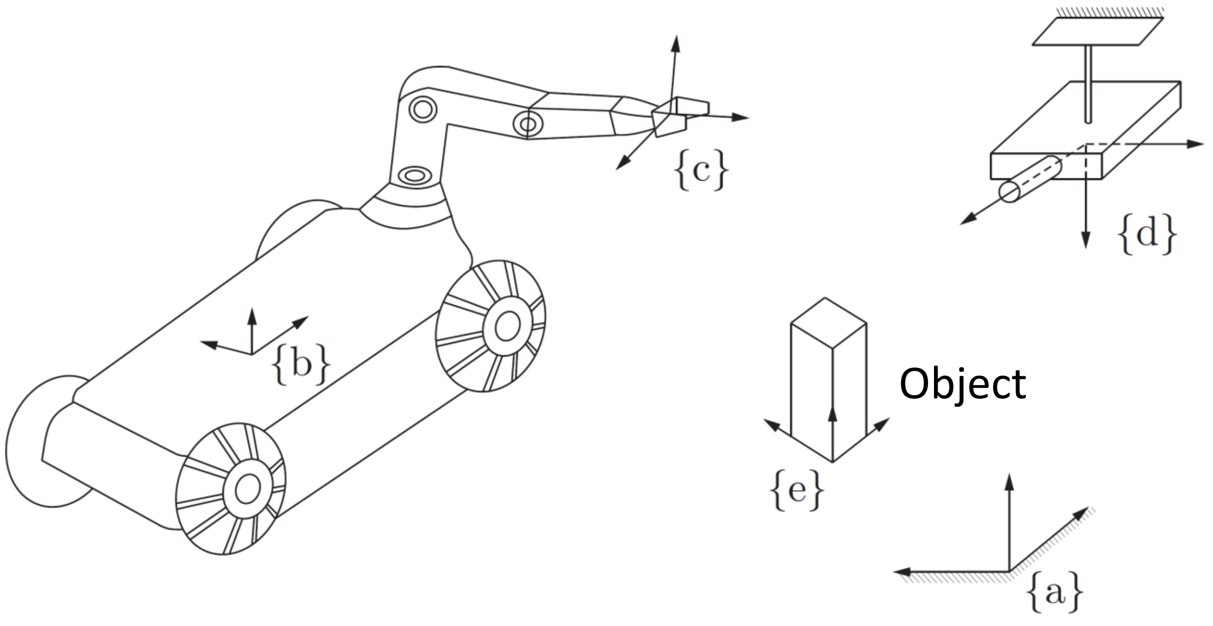
- Forward kinematics (future lectures)



Gripper in robot

$$T_{bc} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 30 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



- We know the following transformations

Robot in camera

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Object in camera

$$T_{de} = \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

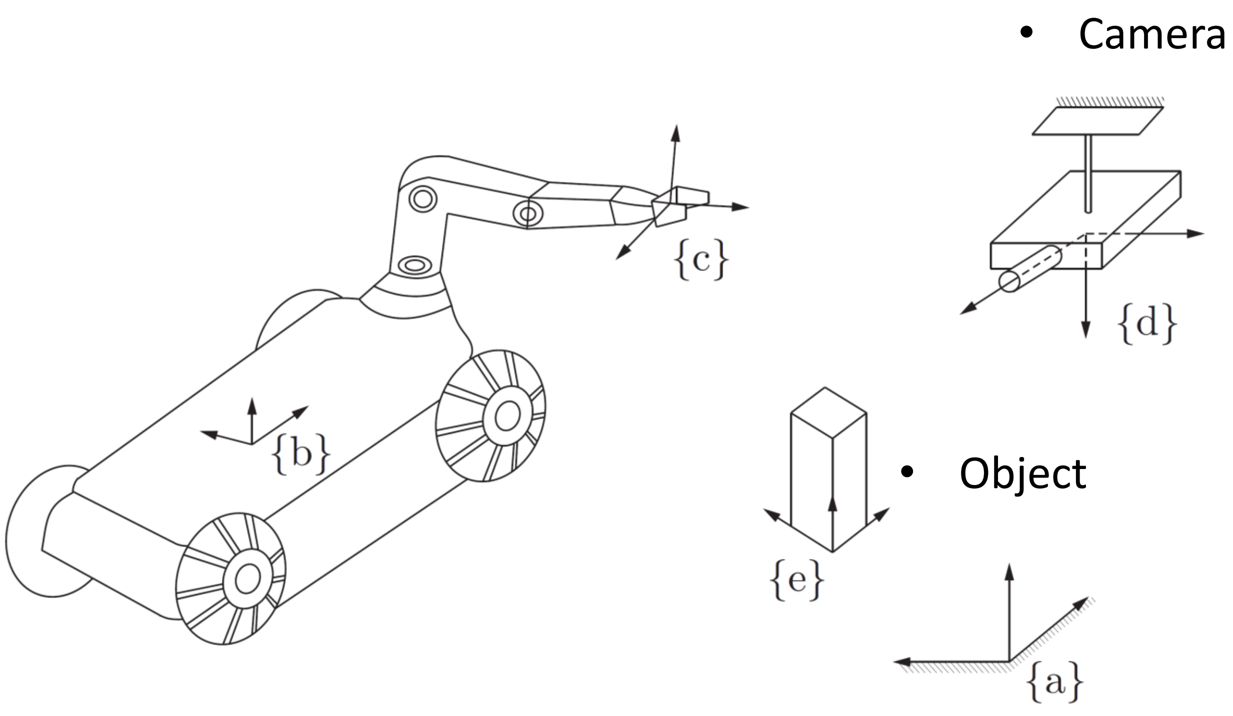
Camera in fixed frame

$$T_{ad} = \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Gripper in robot

$$T_{bc} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 30 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrices



$$T_{ce} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -260/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 130/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- How to move the robot arm to pick up the object?

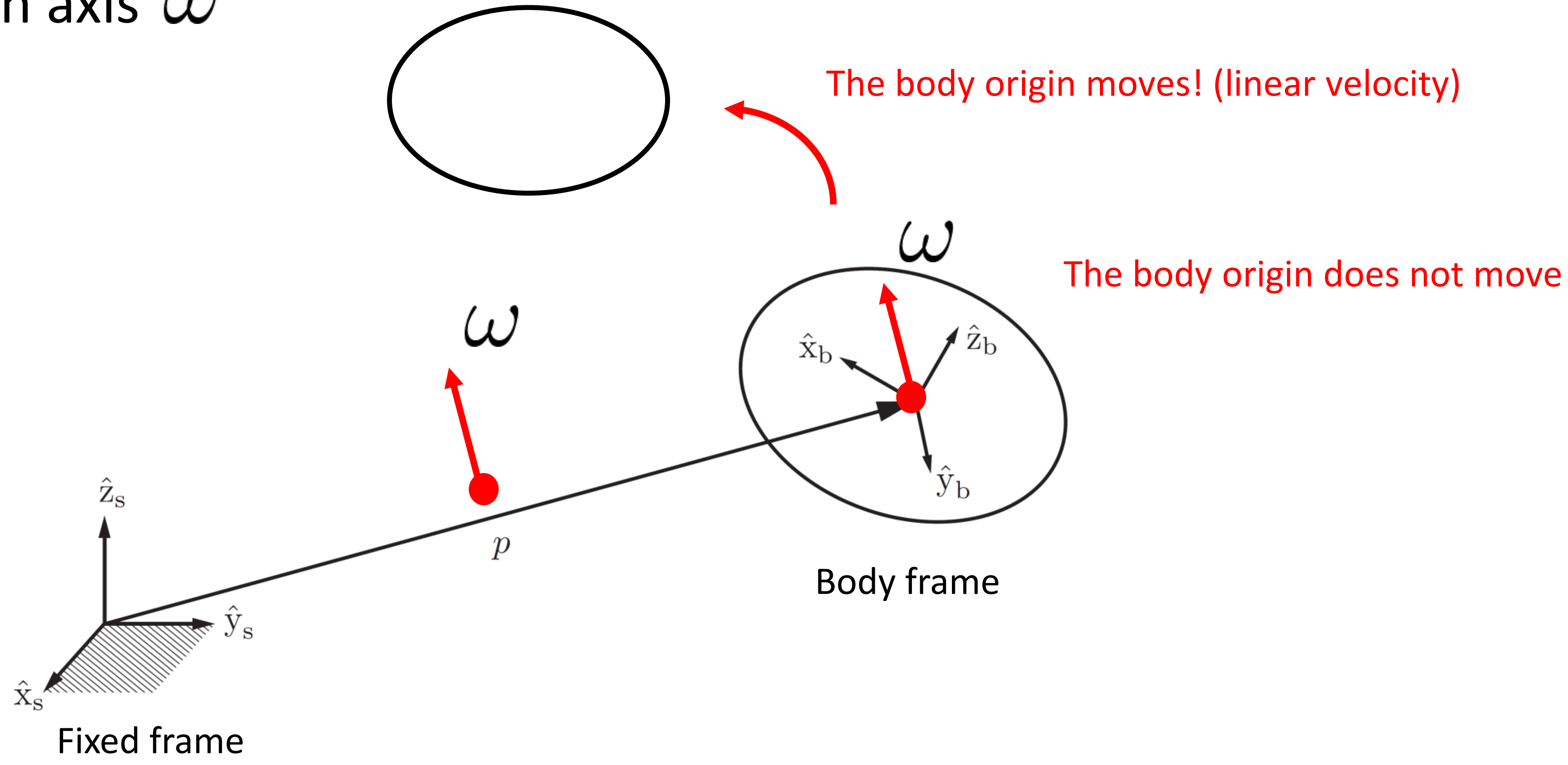
$$T_{ce}$$

- We know T_{db} T_{de} T_{bc} T_{ad}

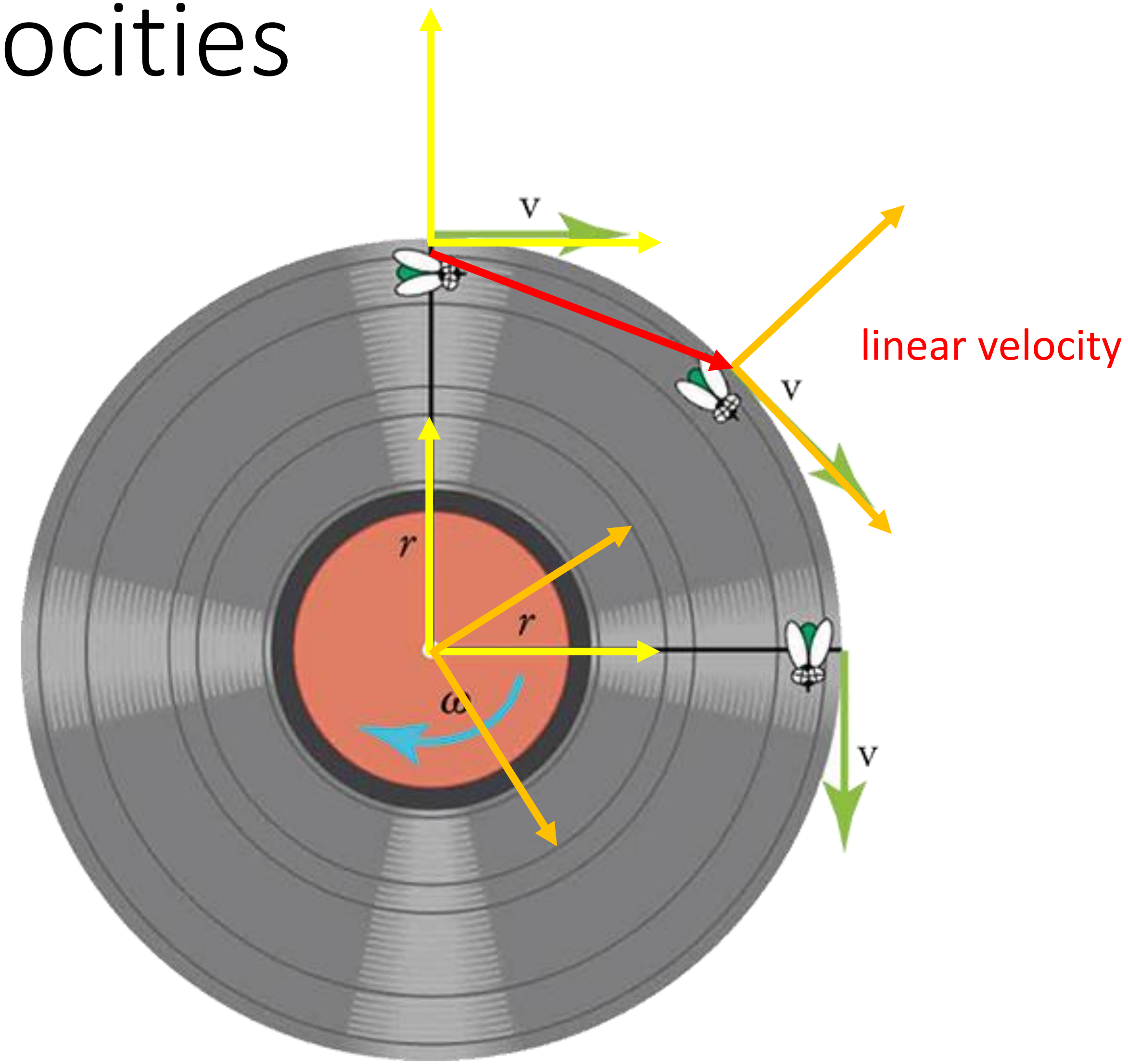
$$T_{ce} = T_{bc}^{-1} T_{db}^{-1} T_{de}$$

Angular Velocities

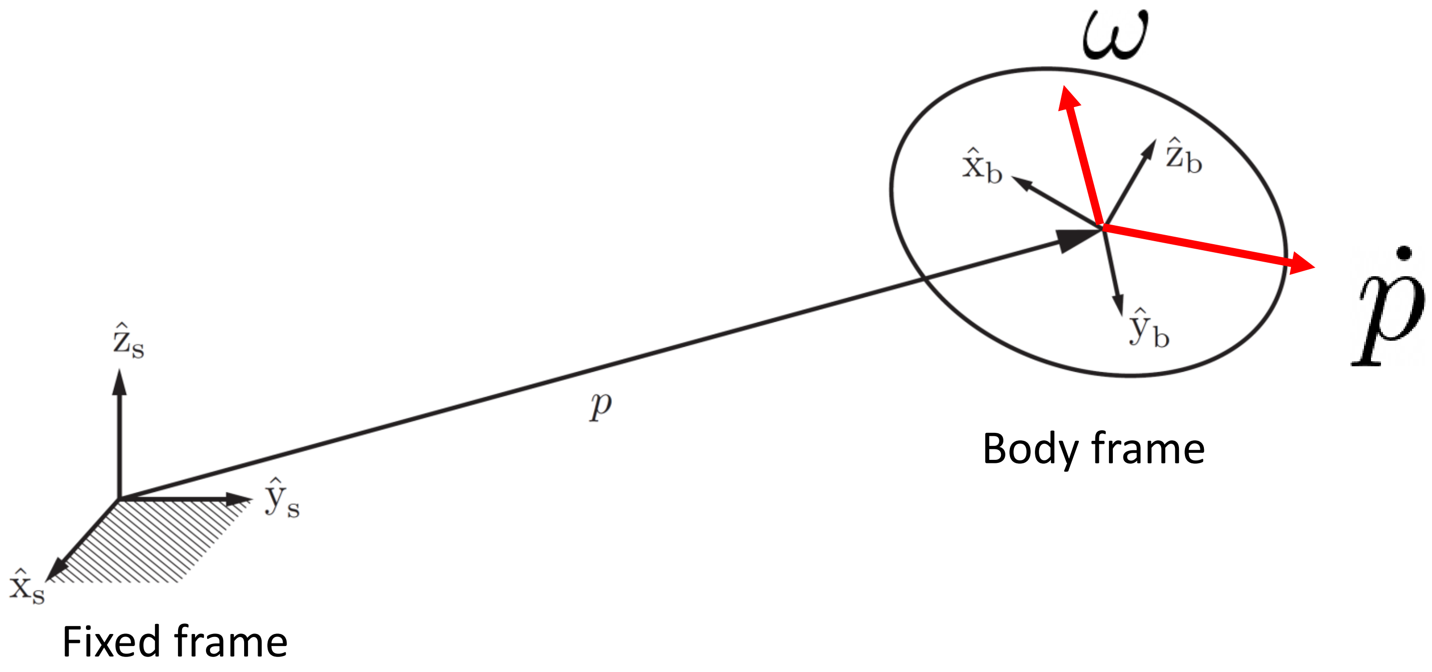
- Rotation axis ω



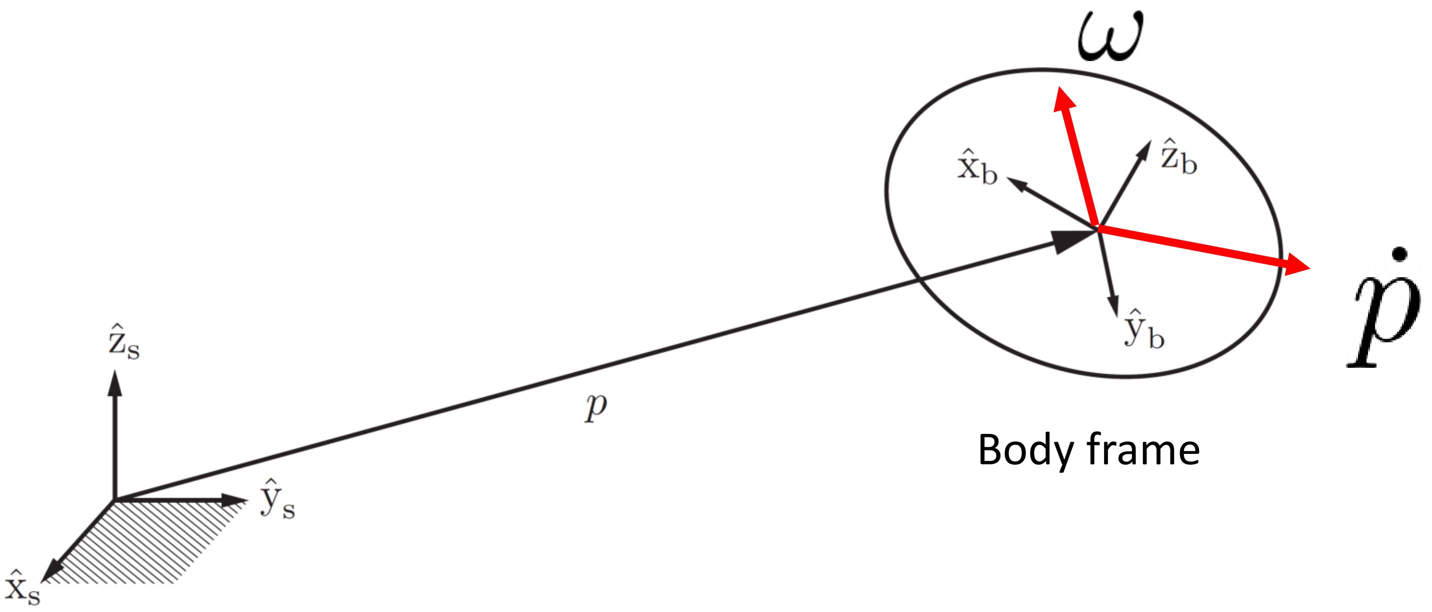
Angular Velocities



Angular Velocity and Linear Velocity



Angular Velocity and Linear Velocity



$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$

$$\dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

- Angular velocity

$$\omega = \hat{\omega}\dot{\theta}$$

$$\dot{R}R^{-1} = [\omega_s]$$

$$R^{-1}\dot{R} = [\omega_b]$$

- Linear velocity $\dot{p}$

The linear velocity of the origin of {b} expressed in the fixed frame {s}

Angular Velocity and Linear Velocity

- Let's compute

$$T^{-1}\dot{T} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} R^{\mathrm{T}}\dot{R} & R^{\mathrm{T}}\dot{p} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}.$$

$$R^{\mathrm{T}}\dot{p} = v_b$$

$$R^T = R_{bs}$$

linear velocity of a point at the origin of {b} expressed in {b}

Recall

$$\dot{R}R^{-1} = [\omega_s]$$
$$R^{-1}\dot{R} = [\omega_b]$$

Skew-symmetric Matrix

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix}$$

Twists: Angular Velocity and Linear Velocity

- Body twist (spatial velocity in the body frame)

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6 \qquad R^{\mathrm{T}}\dot{p} = v_b$$

matrix representation

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$$

$[\omega_b] \in so(3)$ $\quad v_b \in \mathbb{R}^3$ **linear velocity of a point at the origin of {b} expressed in {b}**

Twists

- Similarly

$$\dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{R}R^{\mathrm{T}} & \dot{p} - \dot{R}R^{\mathrm{T}}p \\ 0 & 0 \end{bmatrix}$$

$$[\omega_s] = \dot{R}R^{\mathrm{T}}$$

$$= \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}.$$

$$v_s = \dot{p} - \dot{R}R^{\mathrm{T}}p$$

What is this?

Twists

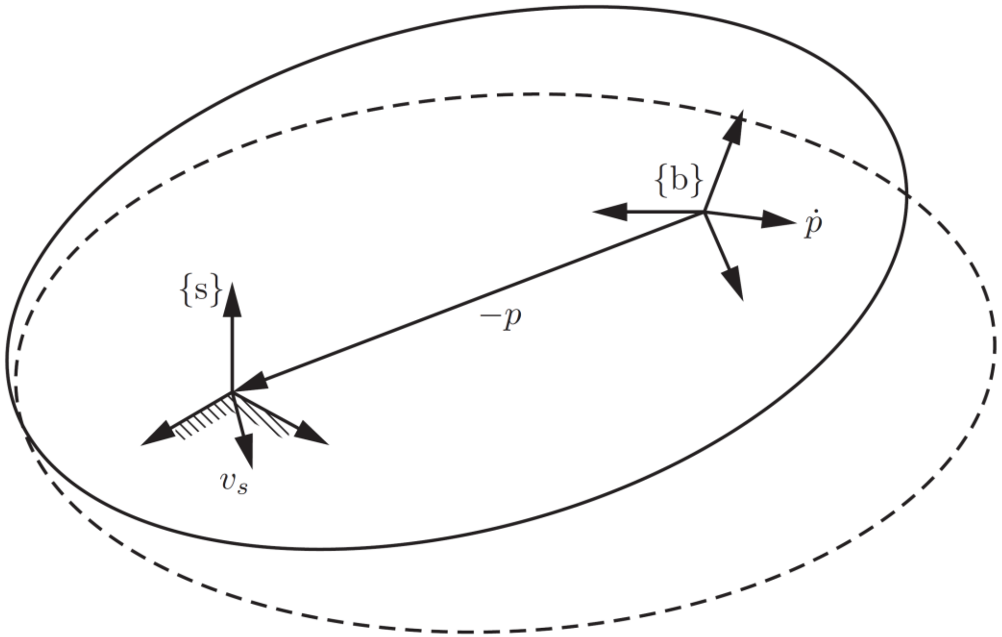
$$v_s = \dot{p} - \dot{R}R^{\mathrm{T}}p$$

Not the linear velocity in fixed frame $\dot{p}$

$$[\omega_s] = \dot{R}R^{\mathrm{T}}$$

$$v_s = \dot{p} - \omega_s \times p = \dot{p} + \omega_s \times (-p)$$

Instantaneous velocity of the point on the rigid body currently at the origin of {s} expressed in {s}



Imagining the moving body to be infinitely large

Tangential Velocity $\dot{p} = \hat{\omega} \times p$

Twists

- Spatial twist (spatial velocity in the space frame)

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6 \qquad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3)$$

- Relationship

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}[\mathcal{V}_s]T \end{aligned} \qquad [\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}$$

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017
- M. Ceccarelli. Screw axis defined by Giulio Mozzi in 1763 and early studies on helicoidal motion. Mechanism and Machine Theory, 35:761-770, 2000.
- J. M. McCarthy. Introduction to Theoretical Kinematics. MIT Press, Cambridge, MA, 1990.