# Matrix Logarithm of Rotations and Homogeneous Transformation Matrices

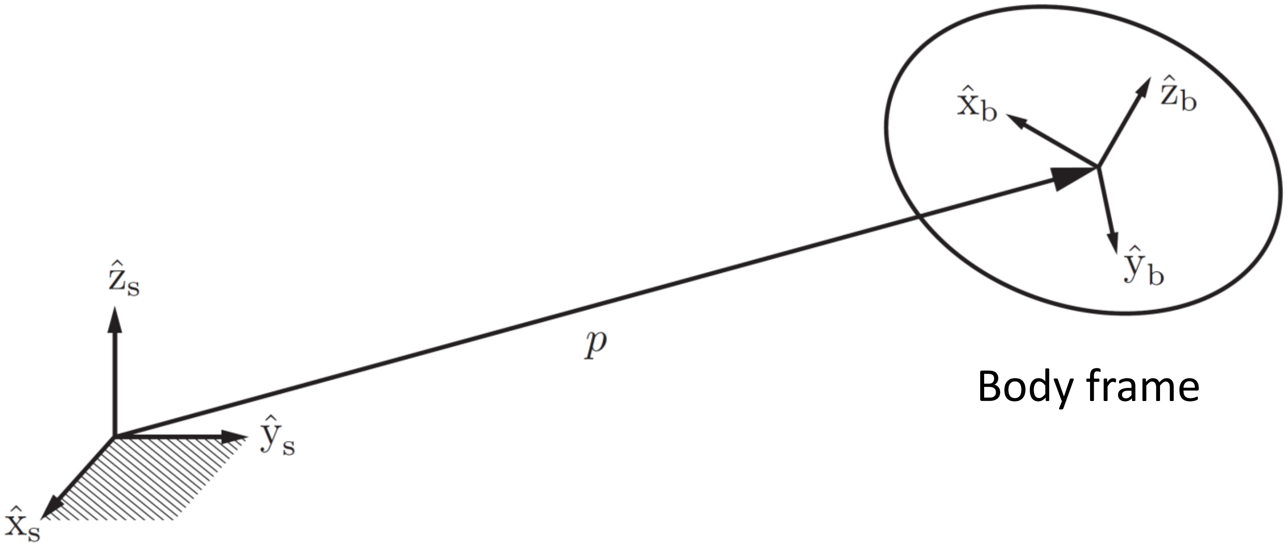
CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

Professor Yu Xiang

The University of Texas at Dallas

# Rigid-Body in 3D

Translation $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$




Body frame

Fixed frame

$$R = [\hat{x}_b \;\; \hat{y}_b \;\; \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Rotation matrix

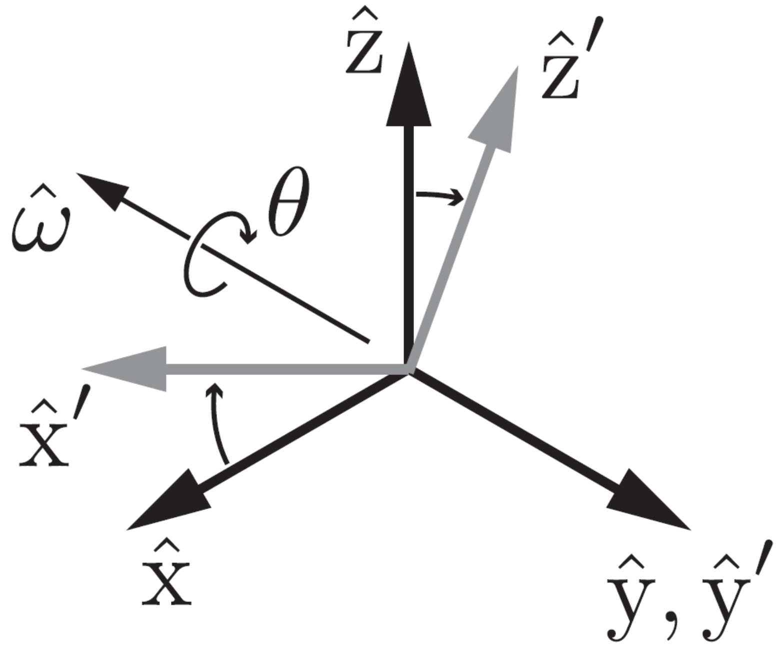
An example

$$p = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Exponential Coordinates of Rotations

- Exponential coordinates
  - A rotation axis (unit length) $\hat{\omega}$
  - An angle of rotation about the axis $\theta$

$$\hat{\omega}\theta \in \mathbb{R}^3$$




Fix the origin when rotating

# Rodrigues' formula

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

Skew-symmetric Matrix

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix}$$

# Matrix Logarithm of Rotations

$$R = e^{[\hat{\omega}]\theta}$$

$$\log(R) = \log(e^{[\hat{\omega}]\theta}) = [\hat{\omega}]\theta$$

- If $\hat{\omega}\theta \in \mathbb{R}^3$ represent the exponential coordinates of rotation R, then the matrix logarithm of the rotation R is

$$[\hat{\omega}\theta] = [\hat{\omega}]\theta$$

# Matrix Logarithm of Rotations

## How to compute matrix logarithm?

$$\text{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3) \quad s_\theta = \sin\theta \quad c_\theta = \cos\theta$$

$$\begin{aligned} r_{32} - r_{23} &= 2\hat{\omega}_1 \sin\theta, \\ r_{13} - r_{31} &= 2\hat{\omega}_2 \sin\theta, \\ r_{21} - r_{12} &= 2\hat{\omega}_3 \sin\theta. \end{aligned}$$

if $\sin\theta \neq 0$ → $\theta \neq k\pi$

$$\begin{aligned} \hat{\omega}_1 &= \frac{1}{2\sin\theta}(r_{32} - r_{23}), \\ \hat{\omega}_2 &= \frac{1}{2\sin\theta}(r_{13} - r_{31}), \\ \hat{\omega}_3 &= \frac{1}{2\sin\theta}(r_{21} - r_{12}). \end{aligned}$$

# Matrix Logarithm of Rotations

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$\operatorname{tr} R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta \qquad \hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$$

When $\theta \neq k\pi$ $\sin\theta \neq 0$ $\theta = \cos^{-1}\left(\frac{1}{2}(\operatorname{tr} R - 1)\right)$

$$\begin{aligned} \hat{\omega}_1 &= \frac{1}{2\sin\theta}(r_{32} - r_{23}), \\ \hat{\omega}_2 &= \frac{1}{2\sin\theta}(r_{13} - r_{31}), \\ \hat{\omega}_3 &= \frac{1}{2\sin\theta}(r_{21} - r_{12}). \end{aligned}$$

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^{\mathrm{T}})$$

# Matrix Logarithm of Rotations

$$\text{tr}\, R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta \qquad \hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$$

When $\theta = k\pi$

- Even k, R=I, $\hat{\omega}$ undefined
- Odd k, $\theta = \pm\pi, \pm 3\pi, \ldots,$ $R = e^{[\hat{\omega}]\pi} = I + 2[\hat{\omega}]^2$

$\text{tr}\, R = -1$ Solve this equation to compute $\hat{\omega}$

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

# Matrix Logarithm of Rotations

- Solutions exist for $\theta \in [0, 2\pi]$

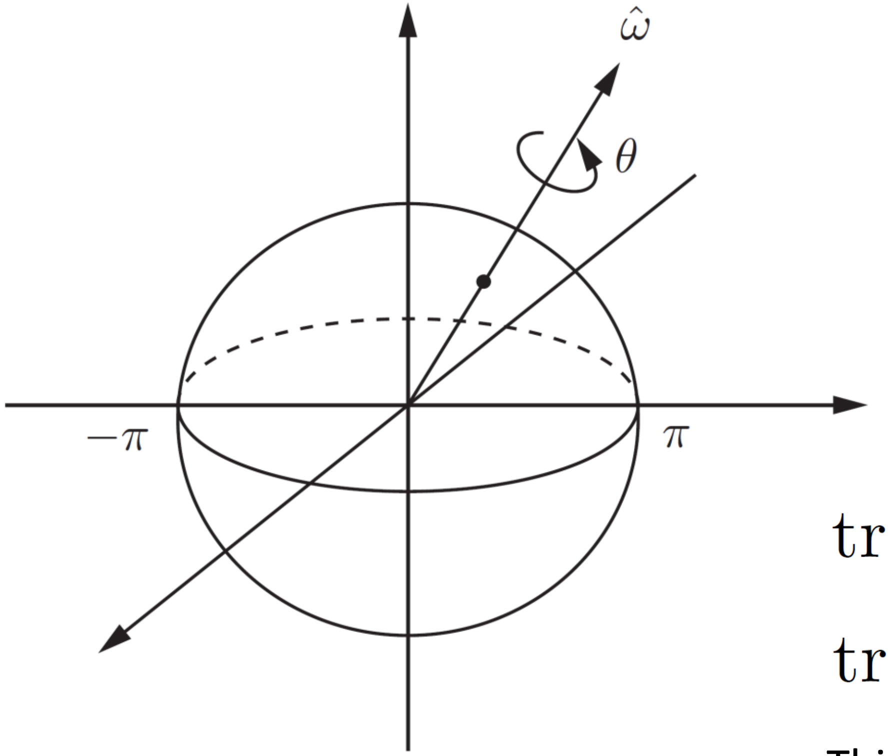
- We can restrict the solution to $\theta \in [0, \pi]$ Why?

$$\mathrm{Rot}(\hat{\omega}, \theta) = \mathrm{Rot}(-\hat{\omega}, -\theta)$$

- See algorithm in Page 74 in Lynch & Park

# Exponential Coordinates and Matrix Logarithm

- Since exponential coordinates $\hat{\omega}\theta$ satisfies $||\hat{\omega}\theta|| \leq \pi$ $\quad \theta \in [0, \pi]$




Rotation axis can take negative direction

$r \in \mathbb{R}^3$ in this solid ball

$$\hat{\omega} = r/\|r\|$$

$$\theta = \|r\| \qquad r = \hat{\omega}\theta$$

$$\operatorname{tr} R \neq -1 \quad e^{[r]} = R \qquad \operatorname{tr} R = 1 + 2\cos\theta$$

$$\operatorname{tr} R = -1 \;\; R = e^{[r]} \text{ with } \|r\| = \pi \quad R = e^{[-r]}$$

This is because $R = e^{[\hat{\omega}]\pi} = I + 2[\hat{\omega}]^2$ and $[\hat{\omega}]^2 = [-\hat{\omega}]^2$

# Exponential Coordinates of Rotations

$$R = e^{[\hat{\omega}]\theta}$$

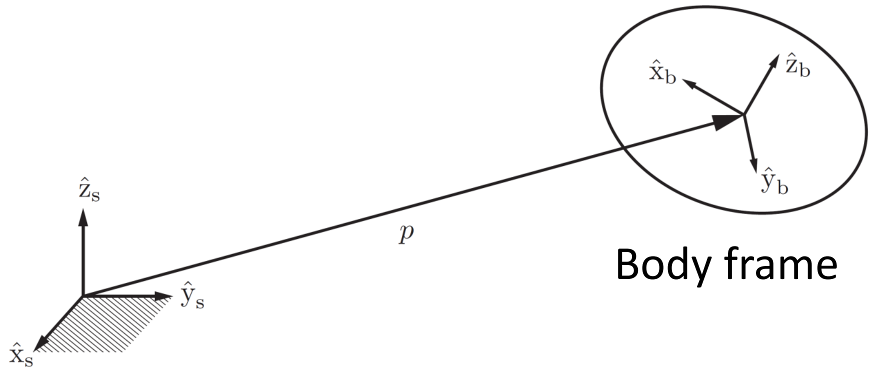
$$[\hat{\omega}] = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix}$$

$$\exp : \quad [\hat{\omega}]\theta \in so(3) \quad \rightarrow \quad R \in SO(3)$$

$$\log : \quad R \in SO(3) \quad \rightarrow \quad [\hat{\omega}]\theta \in so(3)$$

# Homogeneous Transformation Matrices

- Consider body frame {b} in a fixed frame {s}
  - 3D rotation $R \in SO(3)$
  - 3D position $p \in \mathbb{R}^3$




- Special Euclidean group SE(3) or homogenous transformation matrices

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Properties of Transformation Matrices

- The inverse of a transformation matrix is also a transformation matrix

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}} p \\ 0 & 1 \end{bmatrix}$$

- Closure $T_1 T_2$
- Associativity $(T_1 T_2) T_3 = T_1 (T_2 T_3)$
- Identity element: identity matrix $I$
- Not commutative $T_1 T_2 \neq T_2 T_1$

# Homogeneous Coordinates

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

**homogeneous image coordinates**

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = w \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

**homogeneous scene coordinates**

Up to scale

Conversion

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

# Homogeneous Coordinates

$$T\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$$

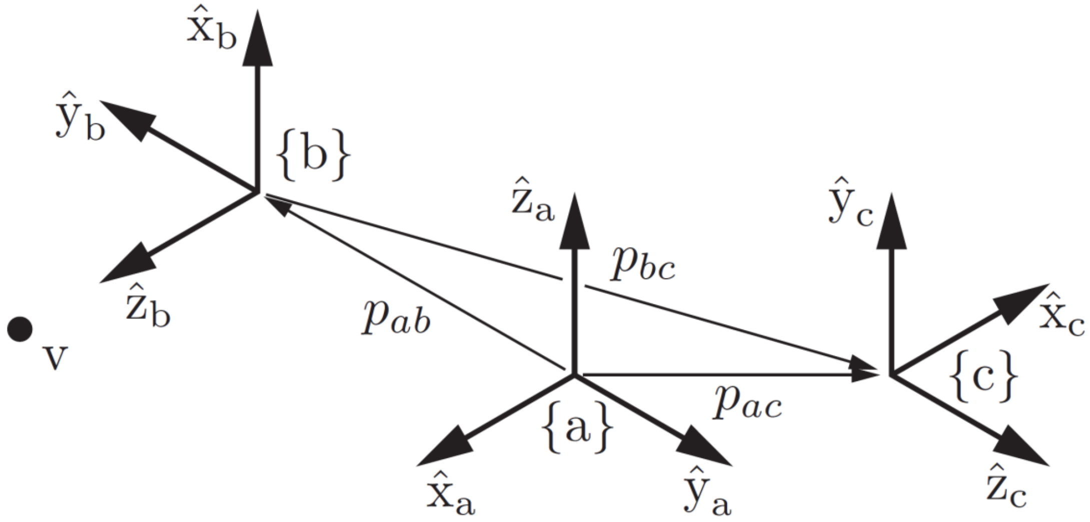
Homogeneous transformation

Homogeneous coordinates

# Uses of Transformation Matrices

- Represent the configuration of a rigid-body
- Change the reference frame
- Displace a vector or a frame

# Representing a Configuration




$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$T_{sa} = (R_{sa}, p_{sa}) \quad T_{sb} = (R_{sb}, p_{sb})$$

$$T_{sc} = (R_{sc}, p_{sc})$$

$$T_{bc} = (R_{bc}, p_{bc}) \quad R_{bc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$p_{bc} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix} \qquad T_{de} = T_{ed}^{-1}$$

# Changing the Reference Frame

$$T_{ab}T_{bc} = T_{a\not{b}}T_{\not{b}c} = T_{ac}$$

$$T_{ab}v_b = T_{a\not{b}}v_{\not{b}} = v_a$$

# Displacing a Vector or a Frame

- Rotating and then translating $(R, p) = (\text{Rot}(\hat{\omega}, \theta), p)$

- Transformation matrices

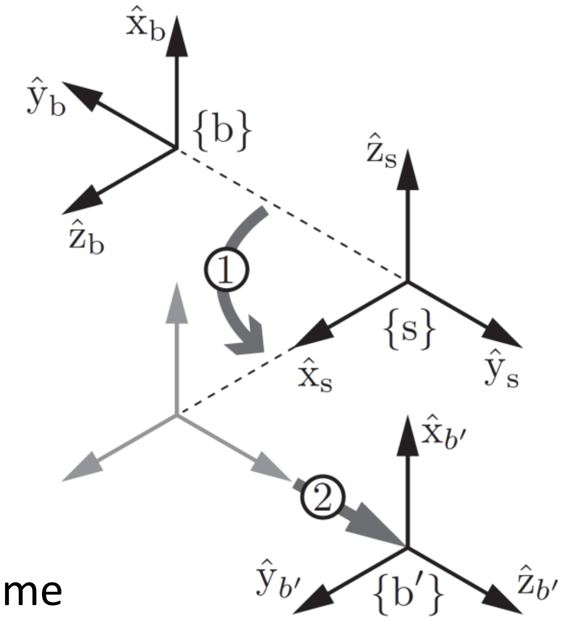
$$\text{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Displacing a Vector or a Frame in Fixed Frame

$$T_{sb'} = TT_{sb} = \text{Trans}(p)\,\text{Rot}(\hat{\omega}, \theta)T_{sb} \qquad \text{(fixed frame)}$$

$$= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & Rp_{sb} + p \\ 0 & 1 \end{bmatrix}$$



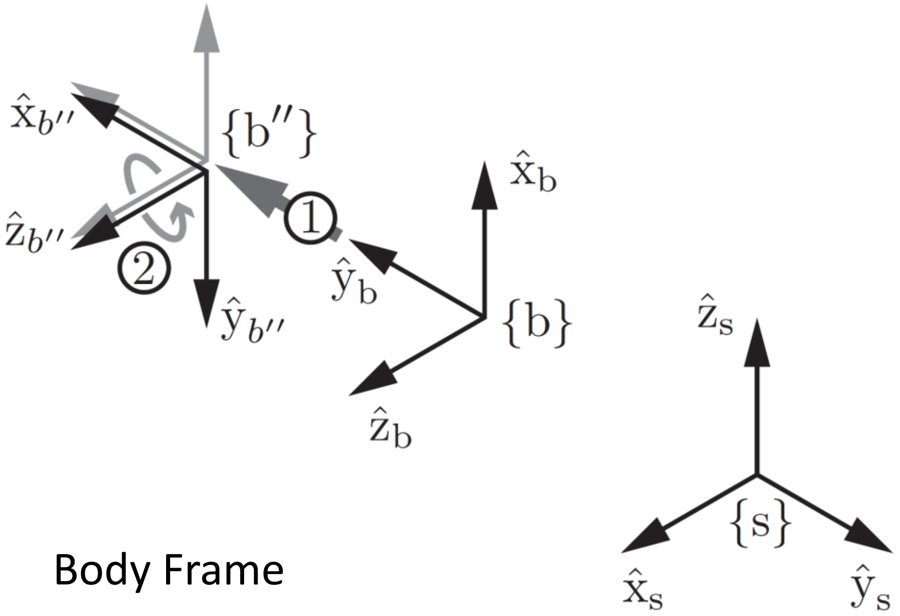

$$\hat{\omega} = (0, 0, 1) \quad \theta = 90^\circ \quad p = (0, 2, 0)$$

Fixed Frame

# Displacing a Vector or a Frame in Body Frame

$$T_{sb''} = T_{sb}T = T_{sb}\,\mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega},\theta) \qquad \text{(body frame)}$$

$$= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \\ 0 & 1 \end{bmatrix}$$




Body Frame

$$\hat{\omega} = (0,0,1) \quad \theta = 90^\circ \quad p = (0,2,0)$$

# Summary

- Matrix Logarithm of Rotations
- Homogenous transformation matrices

# Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017