Angular Velocities and Exponential Coordinates of Rotations

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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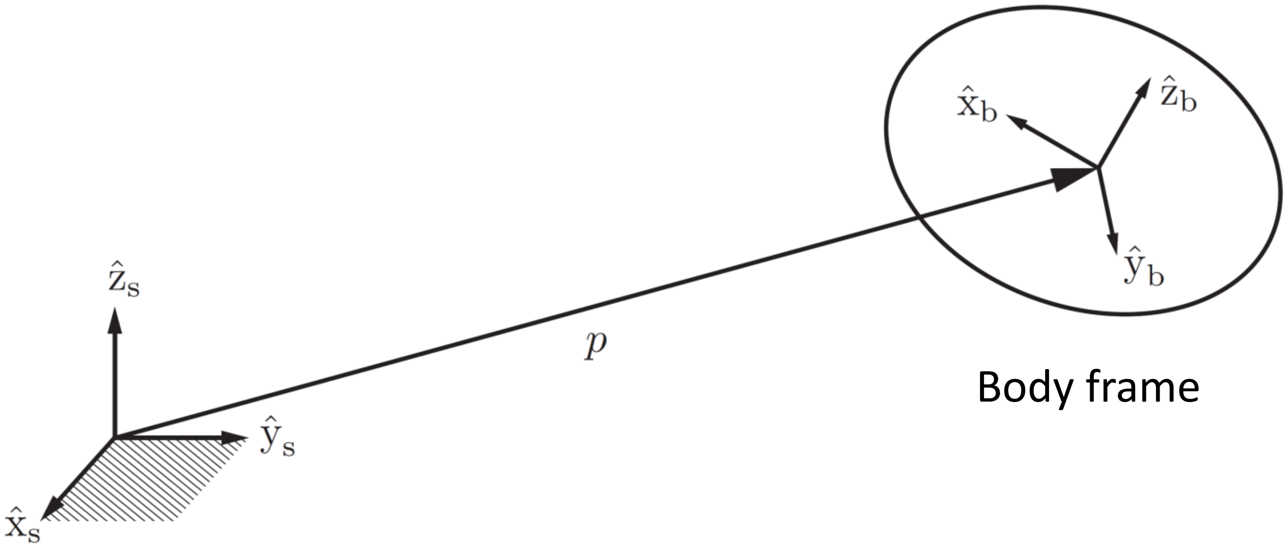
Rigid-Body in 3D



- Origin of the body frame

$$p = p_1\hat{x}_s + p_2\hat{y}_s + p_3\hat{z}_s$$

- Axes of the body frame

$$\begin{aligned}\hat{x}_b &= r_{11}\hat{x}_s + r_{21}\hat{y}_s + r_{31}\hat{z}_s,\\ \hat{y}_b &= r_{12}\hat{x}_s + r_{22}\hat{y}_s + r_{32}\hat{z}_s,\\ \hat{z}_b &= r_{13}\hat{x}_s + r_{23}\hat{y}_s + r_{33}\hat{z}_s.\end{aligned}$$

Translation $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

Rotation matrix

$$R = [\hat{x}_b \;\; \hat{y}_b \;\; \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Meanings of the column vectors

Rotating a Vector or a Frame

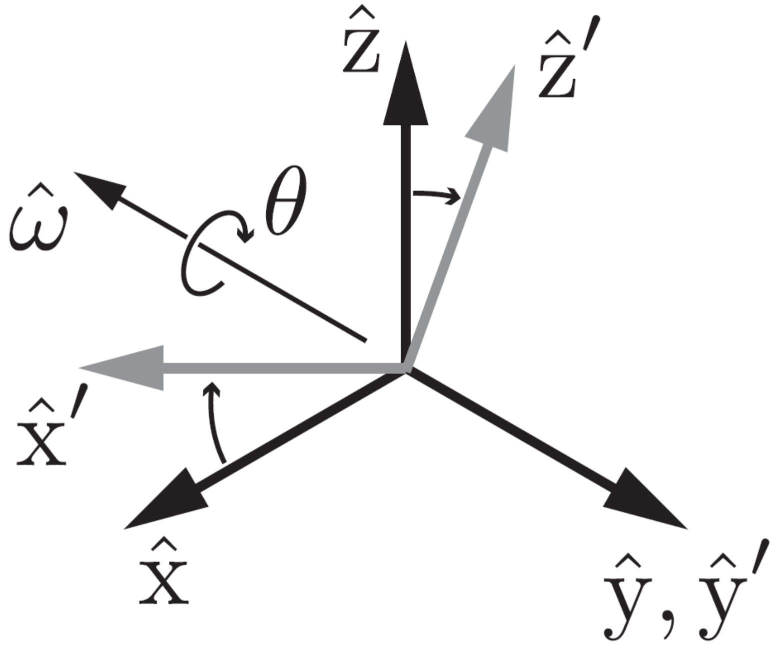
- Rotate frame {c} about a unit axis $\hat{\omega}$ by θ to get frame {c'}, {c} is aligned with {s} in the beginning

$$R = R_{sc'}$$

frame {c'} relative to frame {s}

- Rotation operation

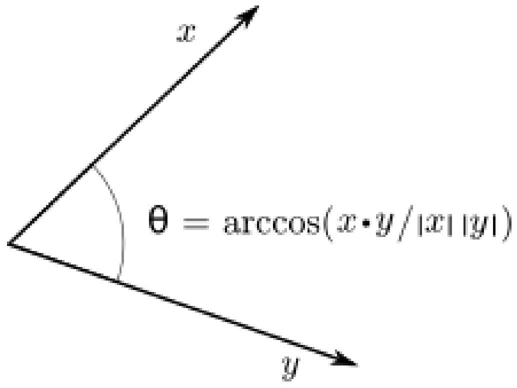
$$R = \mathrm{Rot}(\hat{\omega}, \theta)$$

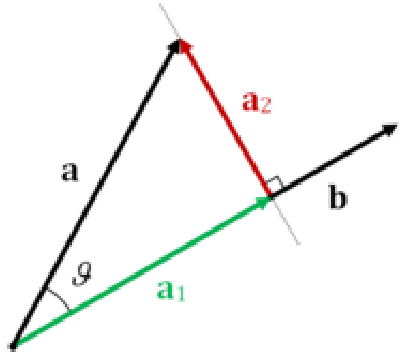


Vector Inner Product

- Dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \ \|\mathbf{b}\| \cos\theta$$





Vector Projection

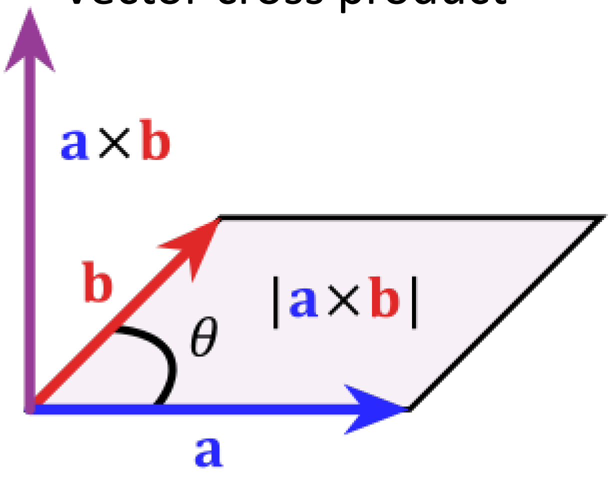
$$a_1 = \|\mathbf{a}\| \cos\theta = \|\mathbf{a}\| \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \ \|\mathbf{b}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|},$$

$$\mathbf{a}_1 = a_1 \hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \frac{\mathbf{b}}{\|\mathbf{b}\|}$$

https://en.wikipedia.org/wiki/Dot_product

Vector Cross Product

Vector cross product



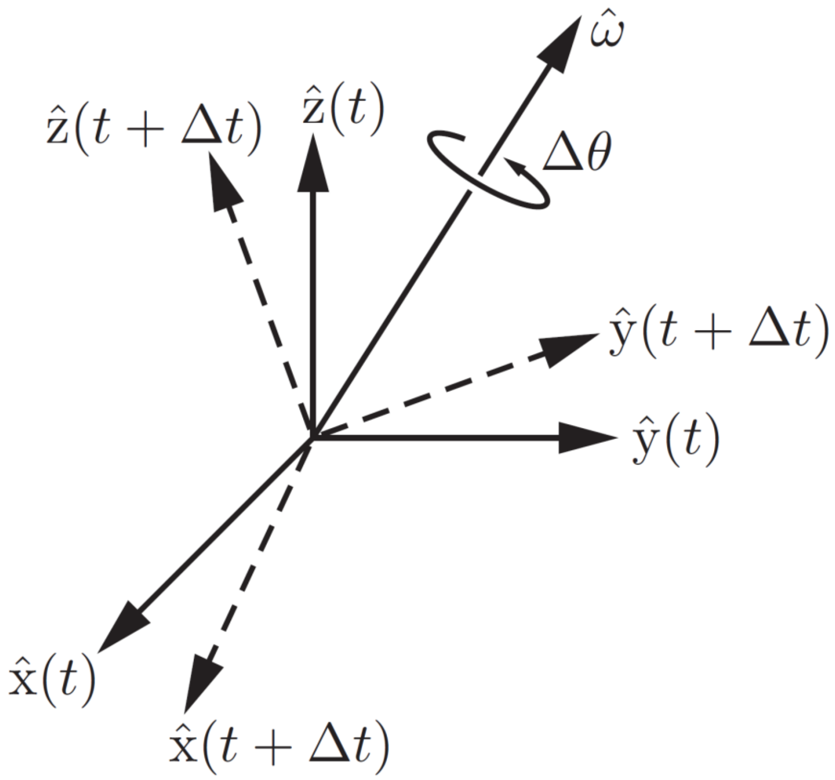
$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \, \mathbf{n}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$
$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

https://en.wikipedia.org/wiki/Cross_product

Angular Velocities



- Axes $\{\hat{x}, \hat{y}, \hat{z}\}$ Unit length

  Rotating around $\hat{\omega}$ by $\Delta\theta$

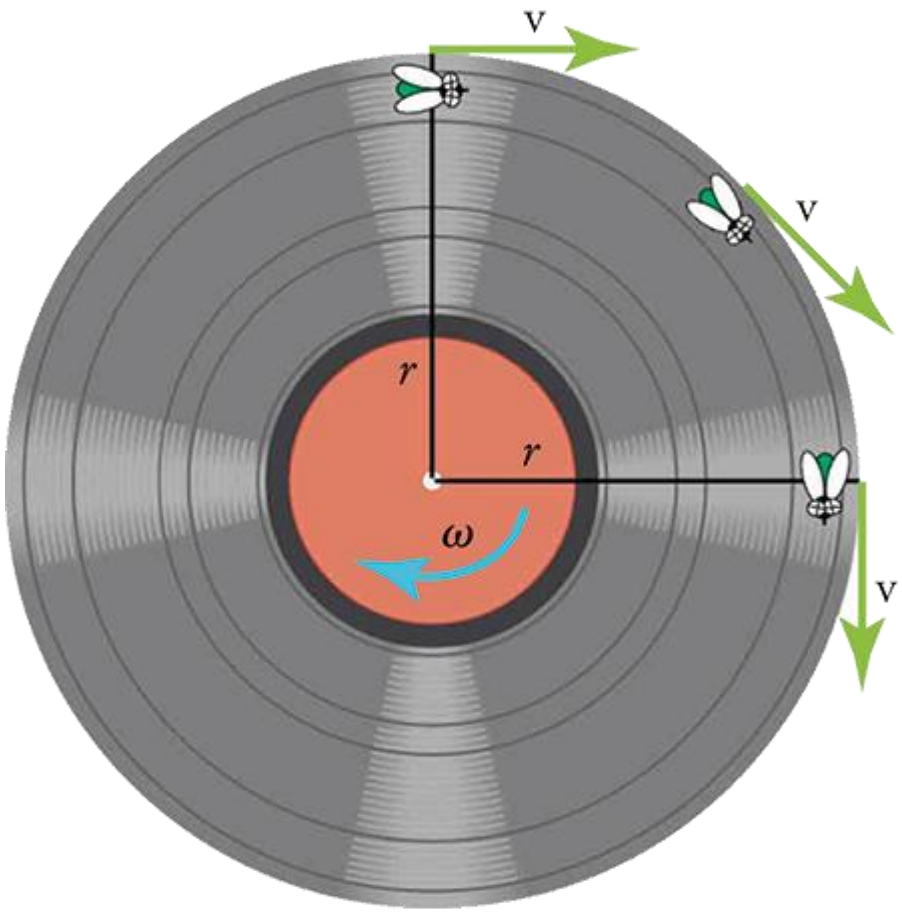
  $\hat{\omega}$ is coordinate free for now

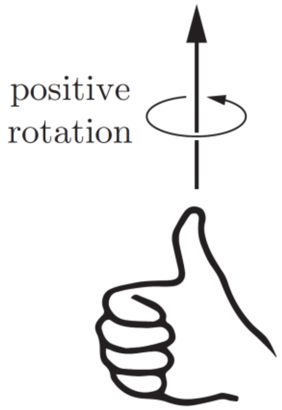
  $\Delta t \rightarrow 0 \quad \Delta\theta / \Delta t \longrightarrow \dot{\theta}$ **Instantaneous velocity**

  $\hat{\omega}$ instantaneous axis of rotation

- **Definition** Angular velocity $\omega = \hat{\omega}\dot{\theta}$

Angular Velocity and Tangential Velocity

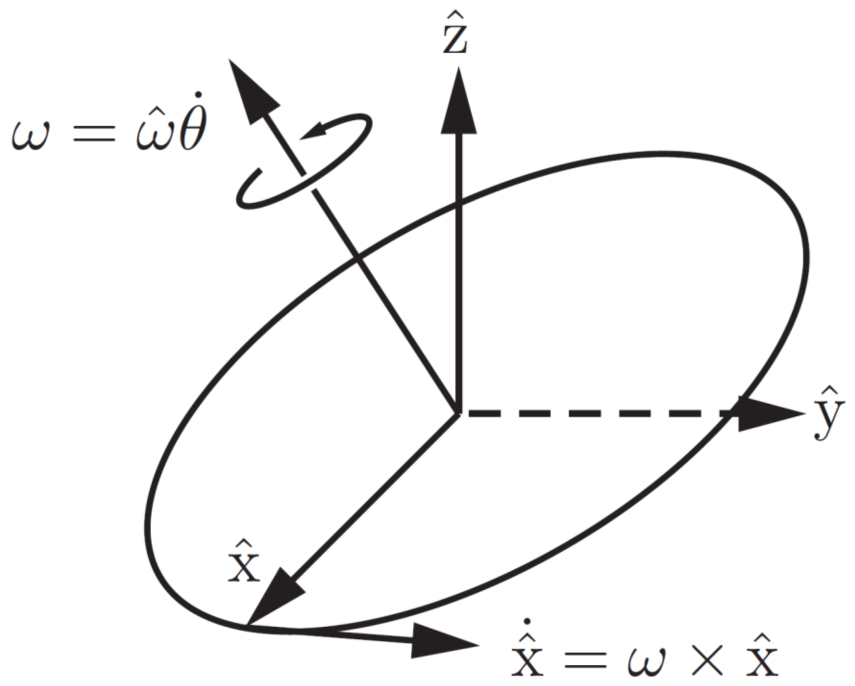




Speed v = r w

https://openstax.org/books/physics/pages/6-1-angle-of-rotation-and-angular-velocity

Angular Velocities



- Angular velocity $\omega = \hat{\omega}\dot{\theta}$
- Compute time derivates of these axes caused by rotation $\dot{\hat{x}}$ (tangential velocity)

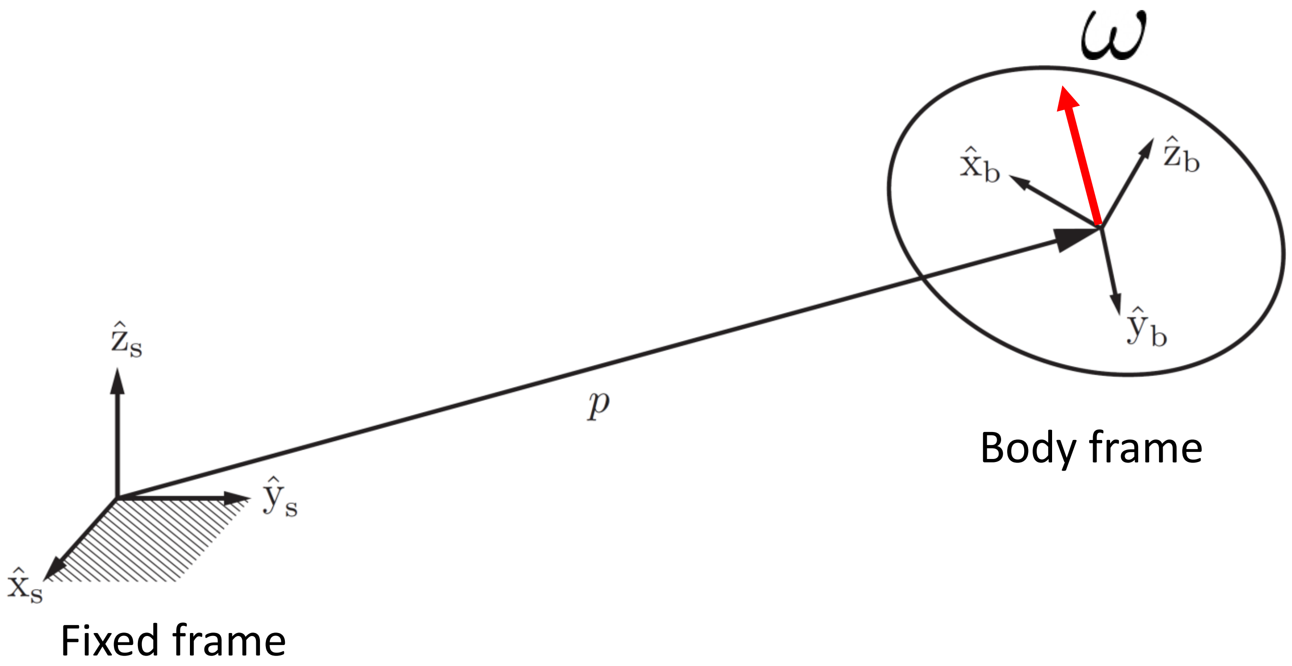
$$\dot{\hat{\mathbf{x}}} = \omega \times \hat{\mathbf{x}}$$

$$\dot{\hat{\mathbf{y}}} = \omega \times \hat{\mathbf{y}}$$

$$\dot{\hat{\mathbf{z}}} = \omega \times \hat{\mathbf{z}}$$

Angular Velocities

- To express these equations in coordinates, we have to choose a reference frame for ω
  - Two natural choices: fixed frame {s} or body frame {b}



Angular Velocities in Fixed Frame

- Consider fixed frame {s}
  - Orientation of the body frame at time t $R(t) = [\hat{\mathrm{x}}_{\mathrm{b}} \;\; \hat{\mathrm{y}}_{\mathrm{b}} \;\; \hat{\mathrm{z}}_{\mathrm{b}}]$
    $= [r_1(t)\; r_2(t)\; r_3(t)]$
  - Time rate of change $\dot{R}(t)$
  - Angular velocity $\omega_s \in \mathbb{R}^3$

$$\begin{aligned} \dot{\hat{\mathrm{x}}} &= \mathrm{w} \times \hat{\mathrm{x}}, \\ \dot{\hat{\mathrm{y}}} &= \mathrm{w} \times \hat{\mathrm{y}}, \\ \dot{\hat{\mathrm{z}}} &= \mathrm{w} \times \hat{\mathrm{z}}. \end{aligned}$$

$$\dot{r}_i = \omega_s \times r_i, \qquad i = 1, 2, 3.$$

Column

$$\dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3] = \omega_s \times R.$$

Skew-symmetric Matrix

https://en.wikipedia.org/wiki/Skew-symmetric_matrix

$$x = [x_1 \ x_2 \ x_3]^{\mathrm{T}} \in \mathbb{R}^3$$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$$[x] = -[x]^{\mathrm{T}}$$

$$\omega_s \times R = [\omega_s]R$$

$$\dot{R} = \omega_s \times R$$

$$[\omega_s]R = \dot{R}$$

$$[\omega_s] = \dot{R}R^{-1}$$

Skew-symmetric Matrix

Proposition $$R[\omega]R^{\mathrm{T}} = [R\omega] \quad \omega \in \mathbb{R}^3 \quad R \in SO(3)$$

See Lynch & Park for proof

Proof. Letting r_i^{T} be the ith row of R, we have

$$\begin{aligned} R[\omega]R^{\mathrm{T}} &= \begin{bmatrix} r_1^{\mathrm{T}}(\omega \times r_1) & r_1^{\mathrm{T}}(\omega \times r_2) & r_1^{\mathrm{T}}(\omega \times r_3) \\ r_2^{\mathrm{T}}(\omega \times r_1) & r_2^{\mathrm{T}}(\omega \times r_2) & r_2^{\mathrm{T}}(\omega \times r_3) \\ r_3^{\mathrm{T}}(\omega \times r_1) & r_3^{\mathrm{T}}(\omega \times r_2) & r_3^{\mathrm{T}}(\omega \times r_3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -r_3^{\mathrm{T}}\omega & r_2^{\mathrm{T}}\omega \\ r_3^{\mathrm{T}}\omega & 0 & -r_1^{\mathrm{T}}\omega \\ -r_2^{\mathrm{T}}\omega & r_1^{\mathrm{T}}\omega & 0 \end{bmatrix} \\ &= [R\omega], \end{aligned}$$

Angular Velocities in Body Frame

- Consider body frame {b} ω_b

Change of reference frame $\omega_s = R_{sb}\omega_b$

$$\omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^{\mathrm{T}}\omega_s$$

$$\begin{aligned}[\omega_b] &= [R^{\mathrm{T}}\omega_s] \\ &= R^{\mathrm{T}}[\omega_s]R \\ &= R^{\mathrm{T}}(\dot{R}R^{\mathrm{T}})R \\ &= R^{\mathrm{T}}\dot{R} = R^{-1}\dot{R}\end{aligned}$$

(proposition) $[\omega_s] = \dot{R}R^{-1}$

Angular Velocities

- Orientation of the body frame at time t in the fixed frame $R(t)$

$$R_{sb}(t)$$

- Angular velocity w

$$\dot{R}R^{-1} = [\omega_s]$$
$$R^{-1}\dot{R} = [\omega_b]$$

- Change of reference frame of angular velocity

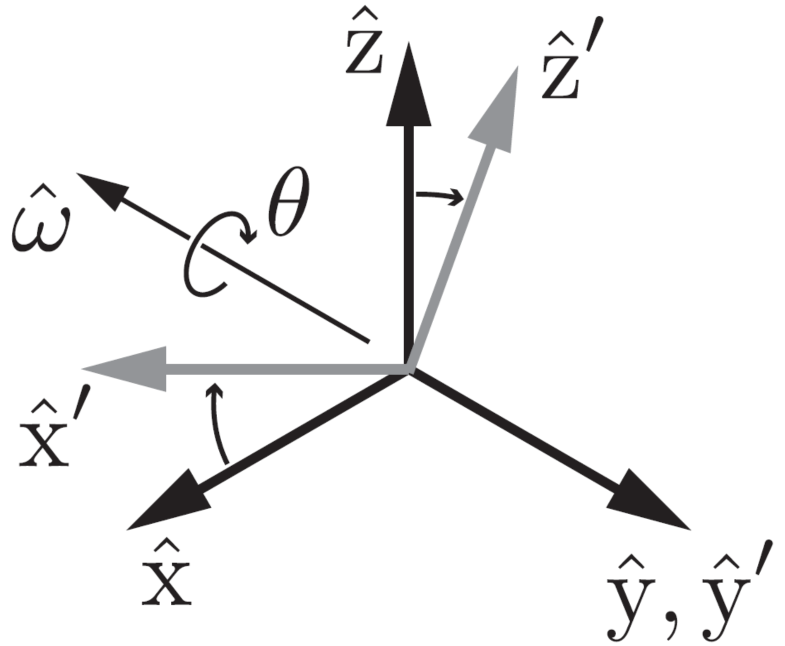
$$\omega_c = R_{cd}\omega_d$$

Exponential Coordinate Representation of Rotation

- Exponential coordinates
  - A rotation axis (unit length) $\hat{\omega}$
  - An angle of rotation about the axis θ

$$\hat{\omega}\theta \in \mathbb{R}^3$$

- Interpretation $R = \mathrm{Rot}(\hat{\omega}, \theta)$
  - Axis-angle rotation of the fixed frame {s}
  - Apply angular velocity $\hat{\omega}\theta$ for one unit of time
  - Apply angular velocity $\hat{\omega}$ for θ units of time



Exponential Coordinates of Rotations

- What is the relationship between

$$\hat{\omega}\theta \in \mathbb{R}^3 \quad \text{and} \quad R = \mathrm{Rot}(\hat{\omega}, \theta)$$

$$\mathrm{Rot}(\hat{\omega}, \theta) =$$

$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta \qquad c_\theta = \cos\theta$$

Let's derive this

Linear Differential Equations

- A differential equation is an equation that relates one or more functions and their derivatives

- A scalar linear differential equation $\dot{x}(t) = ax(t)$ $\quad x(t) \in \mathbb{R},\ a \in \mathbb{R}$

Initial condition $x(0) = x_0$ Solution $x(t) = e^{at}x_0$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

Linear Differential Equations

- Vector linear differential equation

$$\dot{x}(t) = Ax(t) \qquad x(t) \in \mathbb{R}^n, \ A \in \mathbb{R}^{n \times n}$$

Initial condition $x(0) = x_0$ Solution $x(t) = e^{At} x_0$

matrix exponential $$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

If A is constant and finite, this series converges to a finite limit

Exponential Coordinates of Rotations

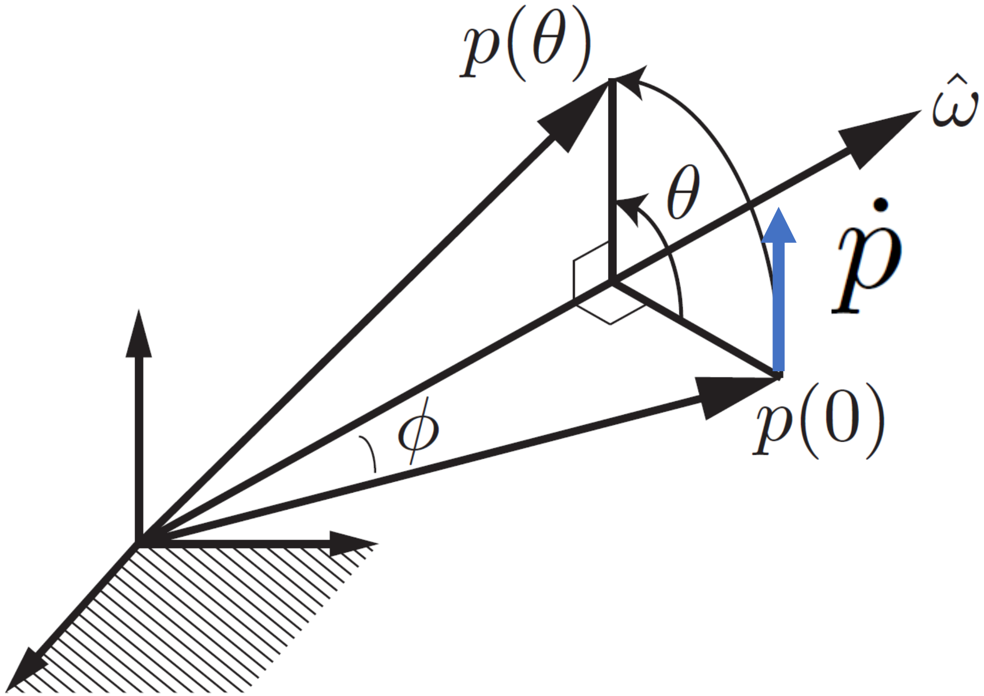
- p(0) is rotated to $p(\theta)$
  - At a constant rate of 1 rad/s

- p(t): path traced by the tip of vector

Tangential Velocity $\dot{p} = \hat{\omega} \times p$

Skew-symmetric Matrix $\dot{p} = [\hat{\omega}]p$

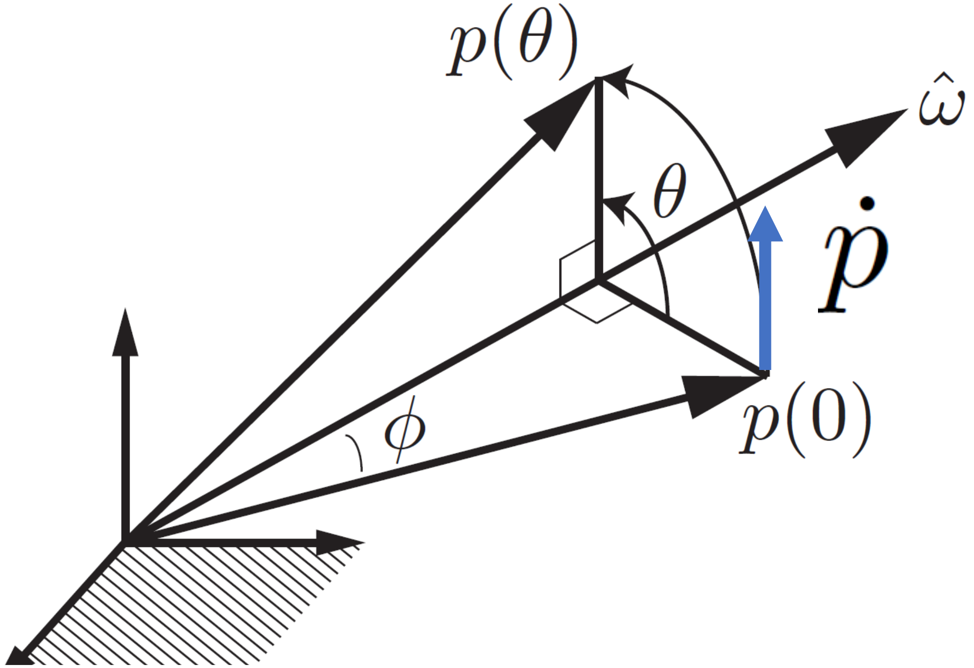
Vector linear differential equations $p(t) = e^{[\hat{\omega}]t} p(0)$



Exponential Coordinates

- $$p(t) = e^{[\hat{\omega}]t} p(0)$$

$$p(\theta) = e^{[\hat{\omega}]\theta} p(0)$$



$$[\hat{\omega}]^3 = -[\hat{\omega}]$$

$$\begin{aligned} e^{[\hat{\omega}]\theta} &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots \\ &= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat{\omega}]^2 \end{aligned}$$

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \end{aligned}$$

$$\text{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

Rodrigues' formula: exponential coordinates to rotation matrix

Rodrigues' formula

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

$$\mathrm{Rot}(\hat{\omega}, \theta) =$$

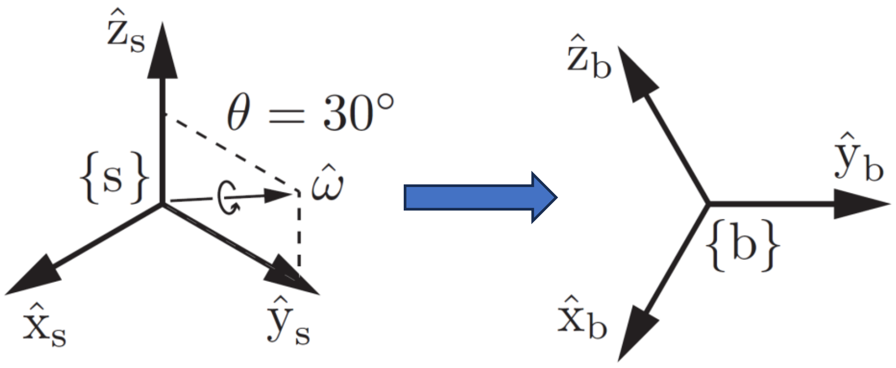
$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta \qquad c_\theta = \cos\theta \qquad \hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$$

Exponential Coordinates of Rotations

- An example

$$\hat{\omega}_1 = (0, 0.866, 0.5) \qquad \theta_1 = 30^\circ$$



$$\begin{aligned} R &= e^{[\hat{\omega}_1]\theta_1} \\ &= I + \sin\theta_1[\hat{\omega}_1] + (1-\cos\theta_1)[\hat{\omega}_1]^2 \\ &= I + 0.5\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix} + 0.134\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0.866 & -0.250 & 0.433 \\ 0.250 & 0.967 & 0.058 \\ -0.433 & 0.058 & 0.899 \end{bmatrix}. \end{aligned}$$

Exponential Coordinates $\hat{\omega}_1\theta_1 = (0, 0.453, 0.262)$

Summary

- Angular velocity

- Exponential coordinates

Further Reading

- Chapter 3 and Appendix B in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017

- Quaternion and Rotations, Yan-Bin Jia, https://graphics.stanford.edu/courses/cs348a-17-winter/Papers/quaternion.pdf

- Introduction to Robotics, Prof. Wei Zhang, OSU, Lecture 3, Rotational Motion, http://www2.ece.ohio-state.edu/~zhang/RoboticsClass/index.html