# Dynamics of a Single Rigid Body and Statics

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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# Robot Dynamics

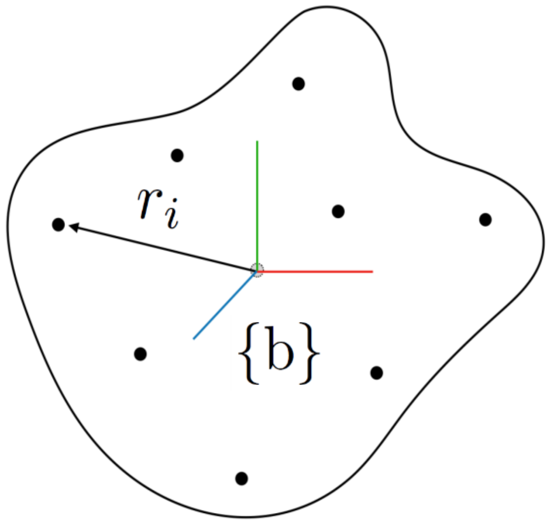
- Study motion of robots with the forces and torques that cause them
  - Newton's second law F = ma

- Forward dynamics
  - Given robot state $(\theta, \dot{\theta})$ and the joint forces and torques $\tau$
  - Determine the robot's acceleration $\ddot{\theta}$

- Inverse dynamics
  - Given robot state $(\theta, \dot{\theta})$ and a desired acceleration $\ddot{\theta}$ (from motion planning)
  - Find the joint forces and torques $\tau$

# Dynamics of a Single Rigid Body

- Assume the body is moving with a body twist $\mathcal{V}_b = (\omega_b, v_b)$
- $p_i(t)$ be the time-varying position of $\mathfrak{m}_i$, initially at $r_i$




$$\dot{p}_i = v_b + \omega_b \times p_i$$

$$\ddot{p}_i = \dot{v}_b + \frac{d}{dt}\omega_b \times p_i + \omega_b \times \frac{d}{dt}p_i$$
$$= \dot{v}_b + \dot{\omega}_b \times p_i + \omega_b \times (v_b + \omega_b \times p_i)$$

$$\ddot{p}_i = \dot{v}_b + [\dot{\omega}_b]r_i + [\omega_b]v_b + [\omega_b]^2 r_i$$

# Dynamics of a Single Rigid Body

- Linear dynamics

Body twist $\mathcal{V}_b = (\omega_b, v_b)$

$$f_b = \mathfrak{m}(\dot{v}_b + [\omega_b]v_b)$$

- Rotational dynamics

Body's rotational inertia matrix

$$m_b = \mathcal{I}_b\dot{\omega}_b + [\omega_b]\mathcal{I}_b\omega_b$$

$$\mathcal{I}_b = -\sum_i \mathfrak{m}_i [r_i]^2 \in \mathbb{R}^{3\times 3}$$

- Rotational kinetic energy

$$\mathcal{K} = \frac{1}{2}\omega_b^{\mathrm{T}}\mathcal{I}_b\omega_b$$

# Twist-Wrench Formulation

- Linear dynamics $f_b = \mathfrak{m}(\dot{v}_b + [\omega_b]v_b)$

- Rotation dynamics $m_b = \mathcal{I}_b\dot{\omega}_b + [\omega_b]\mathcal{I}_b\omega_b$

$$\begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [\omega_b] & 0 \\ 0 & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

# Twist-Wrench Formulation

$$\begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [\omega_b] & 0 \\ 0 & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

Body wrench $\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix}$

Body twist $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$

Spatial inertia matrix $\mathcal{G}_b \in \mathbb{R}^{6\times 6}$ $\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix}$

Spatial momentum $\mathcal{P}_b \in \mathbb{R}^6$ $\mathcal{P}_b = \begin{bmatrix} \mathcal{I}_b\omega_b \\ \mathfrak{m}v_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \mathcal{G}_b\mathcal{V}_b$

# Twist-Wrench Formulation

$$\begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [\omega_b] & 0 \\ 0 & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$\begin{bmatrix} [\omega_b] & 0 \\ 0 & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$= \begin{bmatrix} [\omega_b] & [v_b] \\ 0 & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$[v]v = v \times v = 0 \text{ and } [v]^{\mathrm{T}} = -[v]$$

$$= \begin{bmatrix} [\omega_b] & 0 \\ [v_b] & [\omega_b] \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

# Twist-Wrench Formulation

- Lie bracket of two twists $\mathcal{V}_1 = (\omega_1, v_1)$ and $\mathcal{V}_2 = (\omega_2, v_2)$

$$\begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix} \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = [\mathrm{ad}_{\mathcal{V}_1}]\mathcal{V}_2 = \mathrm{ad}_{\mathcal{V}_1}(\mathcal{V}_2) \in \mathbb{R}^6$$

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$\mathrm{ad}_{\mathcal{V}_1}(\mathcal{V}_2) = -\mathrm{ad}_{\mathcal{V}_2}(\mathcal{V}_1)$$

# Twist-Wrench Formulation

- Dynamic equations for a single rigid body

$$\begin{aligned}\mathcal{F}_b &= \mathcal{G}_b \dot{\mathcal{V}}_b - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_b}(\mathcal{P}_b) \\ &= \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b\end{aligned}$$

- Moment equation for a rigid body

$$m_b = \mathcal{I}_b \dot{\omega}_b - [\omega_b]^{\mathrm{T}} \mathcal{I}_b \omega_b$$

# Dynamics of a Single Rigid Body in Other Frames

- Kinetic energy $= \frac{1}{2}\omega_b^{\mathrm{T}}\mathcal{I}_b\omega_b + \frac{1}{2}\mathfrak{m}v_b^{\mathrm{T}}v_b = \frac{1}{2}\mathcal{V}_b^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b$

- The kinetic energy is independent of frames

$$\begin{aligned}\frac{1}{2}\mathcal{V}_a^{\mathrm{T}}\mathcal{G}_a\mathcal{V}_a &= \frac{1}{2}\mathcal{V}_b^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b \\ &= \frac{1}{2}([\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a)^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a \\ &= \frac{1}{2}\mathcal{V}_a^{\mathrm{T}}\underbrace{[\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]}_{\mathcal{G}_a}\mathcal{V}_a;\end{aligned}$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

# Dynamics of a Single Rigid Body in Other Frames

- The spatial inertia matrix

$$\mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{G}_b [\mathrm{Ad}_{T_{ba}}]$$

- Equations of motion in frame {a}

$$\mathcal{F}_a = \mathcal{G}_a \dot{\mathcal{V}}_a - [\mathrm{ad}_{\mathcal{V}_a}]^{\mathrm{T}} \mathcal{G}_a \mathcal{V}_a$$

# Dynamics of a Single Rigid Body

- Inverse dynamics

$$\mathcal{F}_b \; = \; \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b$$

- Forward dynamics

$$\dot{\mathcal{V}}_b = \mathcal{G}_b^{-1}(\mathcal{F}_b + [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b)$$

# Spatial Force or Wrench

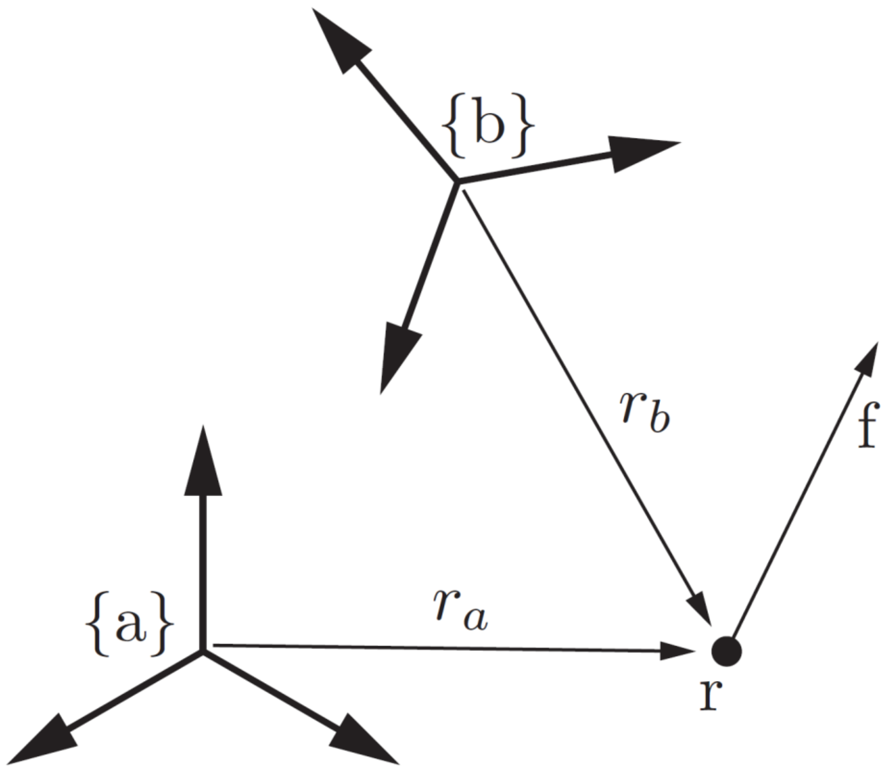
- Merge moment and force in frame {a}

Wrench $$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix} \in \mathbb{R}^6$$

- If more than one wrenches act on a rigid body, the total wrench is the vector sum of the wrenches

- Power = force × velocity $P = Fv$

# Wrench in Different Frames




- Power generated by (F, V) are the same

$$\mathcal{V}_b^{\mathrm{T}} \mathcal{F}_b = \mathcal{V}_a^{\mathrm{T}} \mathcal{F}_a$$

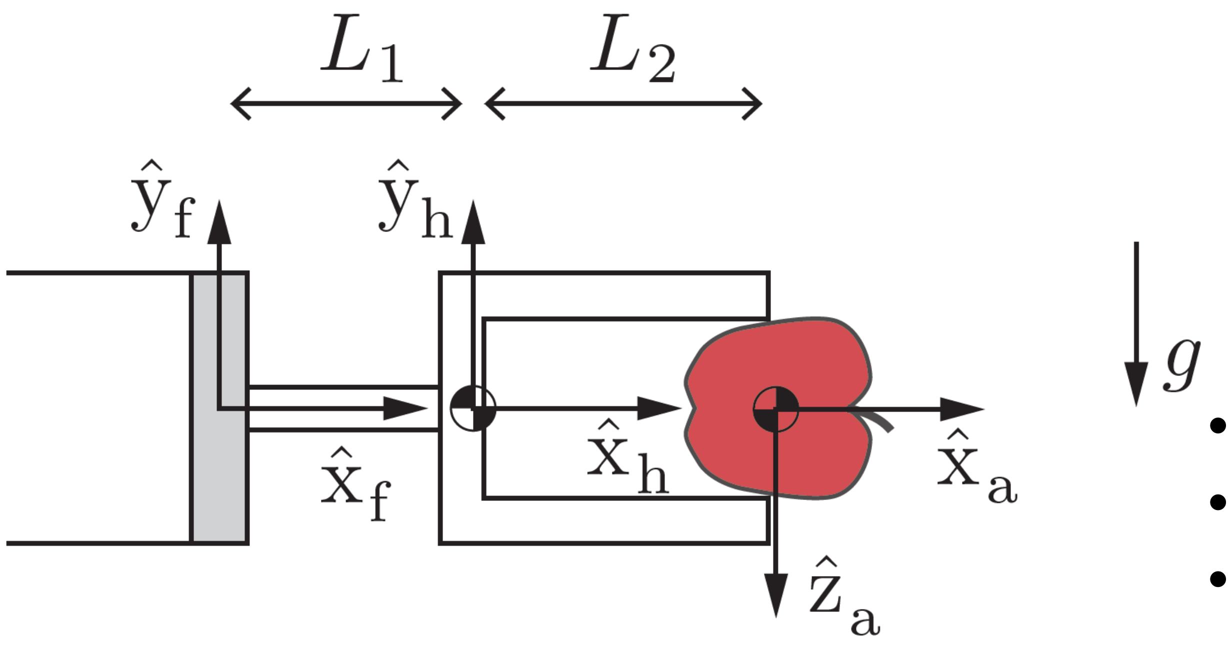
$$\mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}] \mathcal{V}_b$$

$$\begin{aligned} \mathcal{V}_b^{\mathrm{T}} \mathcal{F}_b &= ([\mathrm{Ad}_{T_{ab}}] \mathcal{V}_b)^{\mathrm{T}} \mathcal{F}_a \\ &= \mathcal{V}_b^{\mathrm{T}} [\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}} \mathcal{F}_a. \end{aligned}$$

$$\mathcal{F}_b = [\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}} \mathcal{F}_a$$

$$\mathcal{F}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{F}_b$$

# Wrench Example




A robot hand holding an apple subject to gravity

- Apple mass 0.1 kg
- Gravity g=10 $\mathrm{m/s^2}$
- Mass of hand 0.5 kg

What is the force and torque measured by the six-axis force-torque sensor between the hand and the robot arm?

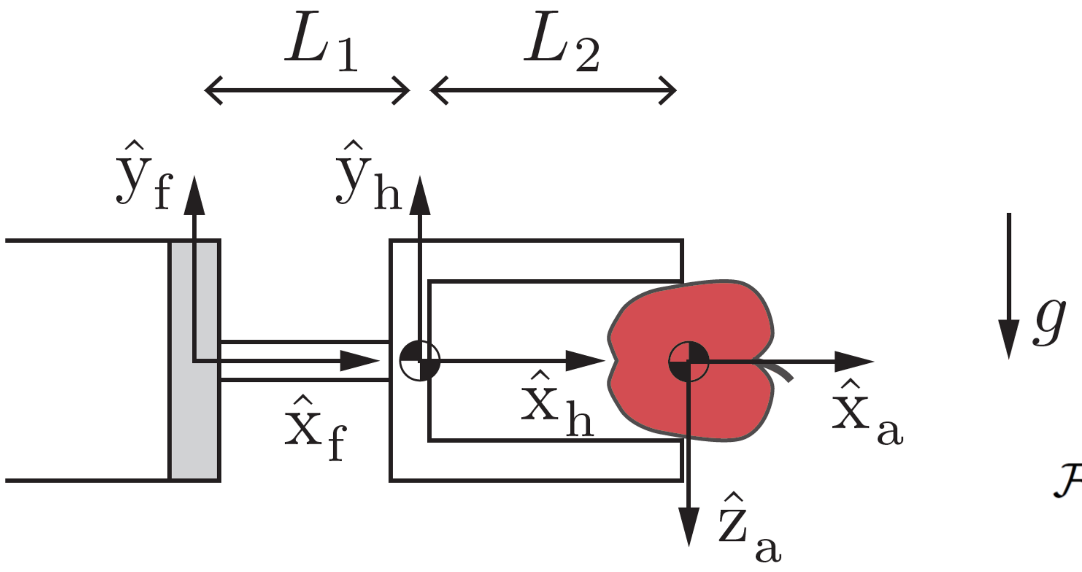
- Frame {f} at the sensor
- Frame {h} at the center of mass of hand
- Frame {a} at the center of mass of apple
- Gravitational wrench on hand in {h}

$$\mathcal{F}_h = (0, 0, 0, 0, -5\ \mathrm{N}, 0)$$

- Gravitational wrench on apple in {a}

$$\mathcal{F}_a = (0, 0, 0, 0, 0, 1\ \mathrm{N})$$

# Wrench Example




A robot hand holding an apple subject to gravity

$$L_1 = 10 \text{ cm} \qquad L_2 = 15 \text{ cm}$$

$$T_{hf} = \begin{bmatrix} 1 & 0 & 0 & -0.1 \text{ m} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_{af} = \begin{bmatrix} 1 & 0 & 0 & -0.25 \text{ m} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \mathcal{F}_f &= [\mathrm{Ad}_{T_{hf}}]^{\mathrm{T}} \mathcal{F}_h + [\mathrm{Ad}_{T_{af}}]^{\mathrm{T}} \mathcal{F}_a \\ &= [0\ 0\ -0.5 \text{ Nm } 0\ -5 \text{ N } 0]^{\mathrm{T}} + [0\ 0\ -0.25 \text{ Nm } 0\ -1 \text{ N } 0]^{\mathrm{T}} \\ &= [0\ 0\ -0.75 \text{ Nm } 0\ -6 \text{ N } 0]^{\mathrm{T}}. \end{aligned}$$

# Statics of Open Chains

- Principle of conservation of power

power at the joints = (power to move the robot) + (power at the end-effector)

- Considering the robot to be at static equilibrium (no power to move robot)

$$\tau^{\mathrm{T}}\dot{\theta} = \mathcal{F}_b^{\mathrm{T}}\mathcal{V}_b$$

power at the end-effector

$$\mathcal{V}_b = J_b(\theta)\dot{\theta}$$

$$\tau = J_b^{\mathrm{T}}(\theta)\mathcal{F}_b$$

# Statics of Open Chains

- If an external wrench $-\mathcal{F}$ is applied to the end-effector when the robot is at equilibrium, joint torque to keep the robot at equilibrium

$$\tau = J^{\mathrm{T}}(\theta)\mathcal{F}$$

- Important for force control

# Summary

- Dynamics of a single rigid body

- Wrench in different frames

- Statics of open chains

# Further Reading

- Sections 3.4, 4.3 and Chapter 8 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.