Forward Kinematics and Product of Exponentials Formula

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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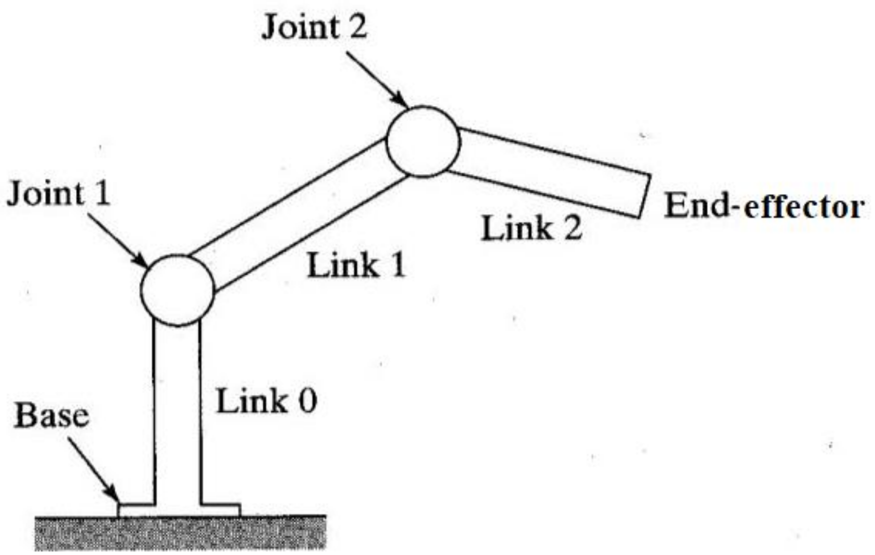
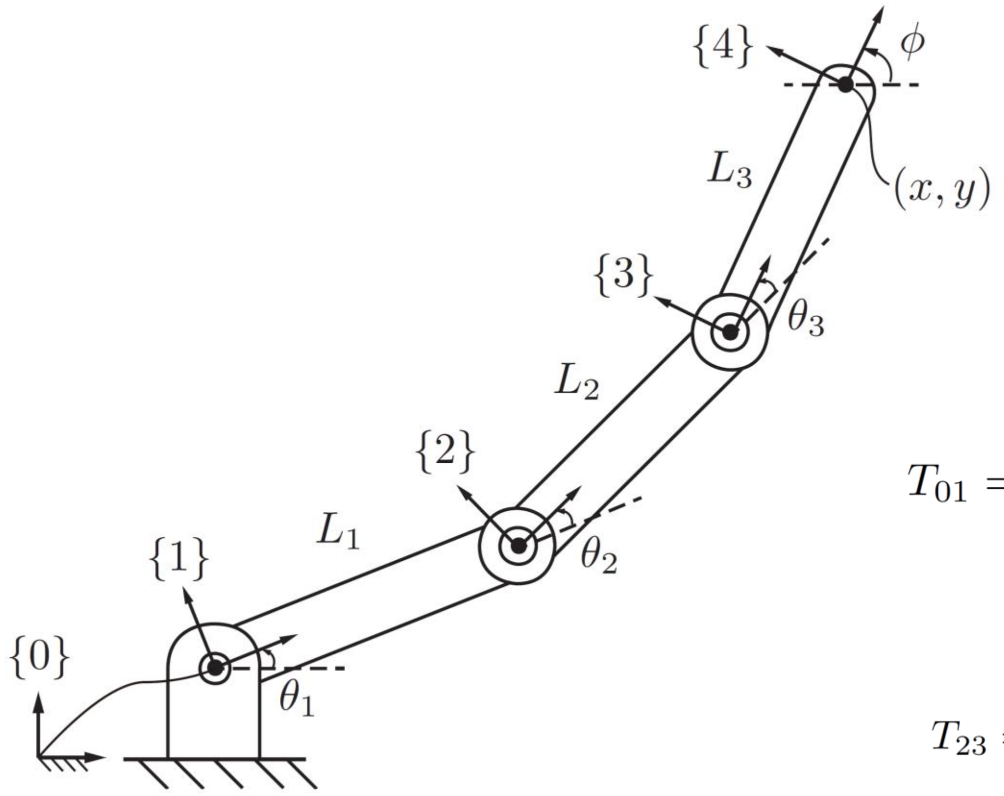
Forward Kinematics

- Forward kinematics of a robot: calculation of the position and orientation of its end-effector from its joint coordinates θ

- Recall robot links and joints



Forward Kinematics



Forward kinematics of a 3R planar open chain.

- General cases
  - Attaching frames to links
  - Using homogeneous transformations

$$T_{04} = T_{01}T_{12}T_{23}T_{34}$$

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$T_{i-1,i}$ Depends only on the joint variable θ_i

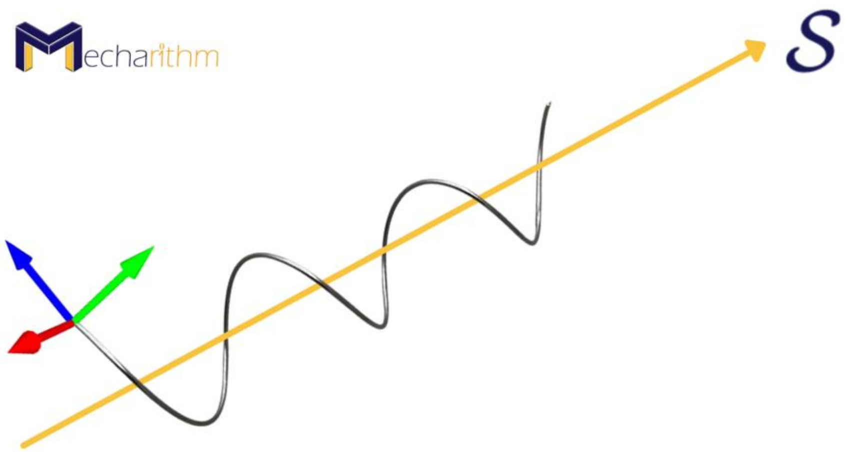
Forward Kinematics

- Method 1: uses homogeneous transformations
  - Need to define the coordinates of frames
  - Denavit-Hartenberg Parameters

- Method 2: uses screw-axis representations of transformations
  - No need to define frame references

Screw-Axis Representations

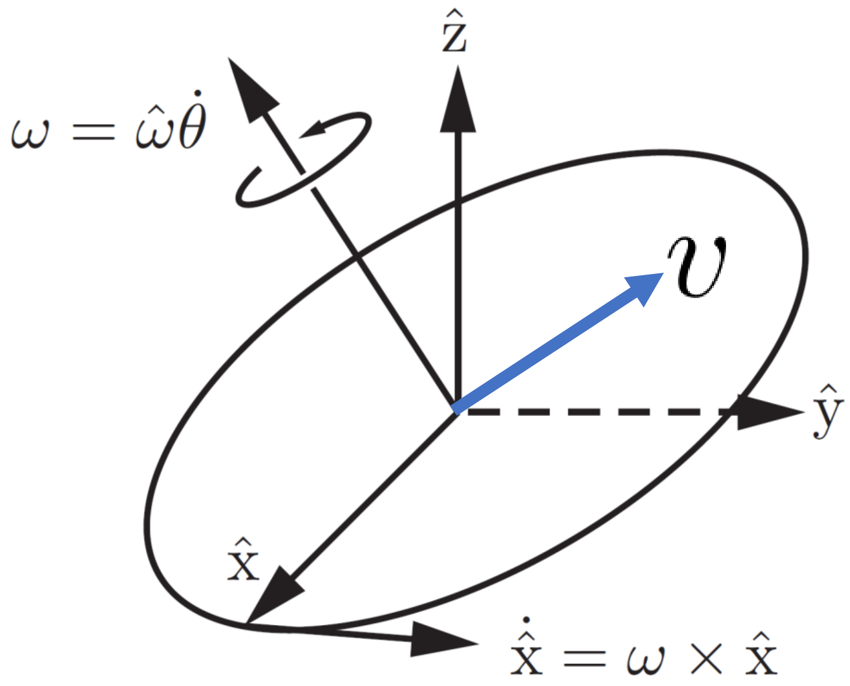
- Screw axis: motion of a screw



https://mecharithm.com/learning/lesson/screw-motion-and-exponential-coordinates-of-robot-motions-11

- **Chasles-Mozzi theorem**: every rigid-body displacement can be expressed as displacement along a fixed screw axis S in space

Exponential Coordinates of Rigid-Body Motions



Velocity of a 3D point

$$\dot{p} = v + \omega \times p$$

$$\dot{\tilde{p}} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \tilde{p}$$

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

$$\tilde{p}(t) = e^{[\mathcal{V}]t} \tilde{p}(0)$$

Exponential Coordinates of Rigid-Body Motions

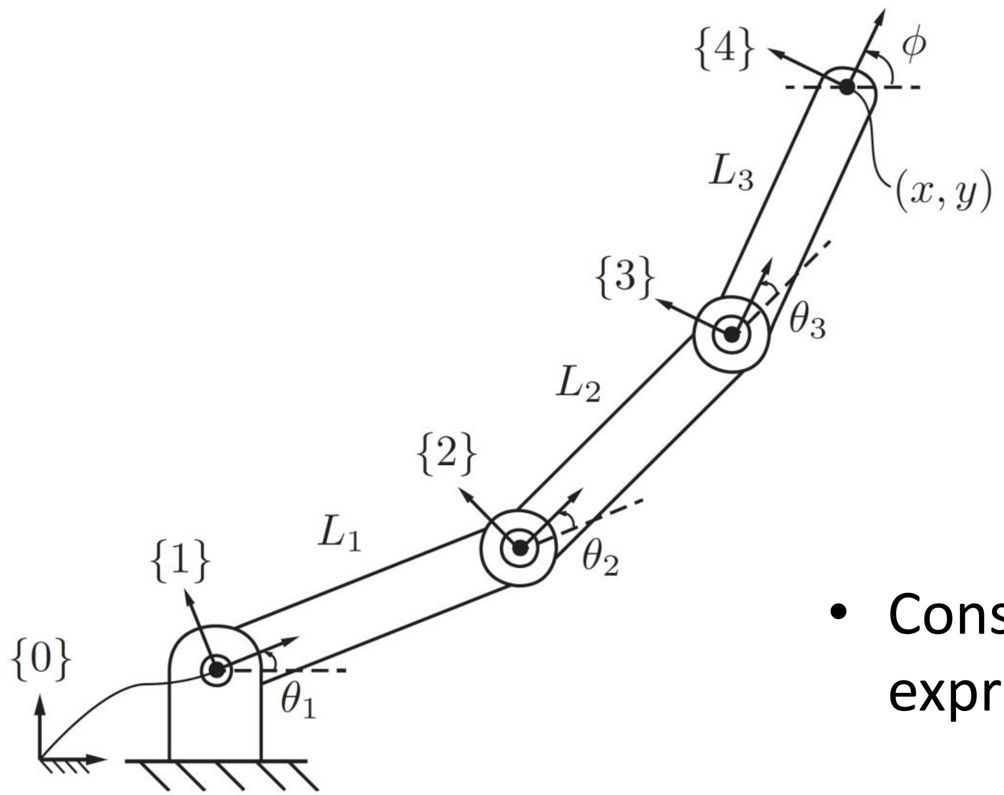
- Exponential coordinates of a homogeneous transformation T

$$\exp: \quad [\mathcal{S}]\theta \in se(3) \quad \rightarrow \quad T \in SE(3)$$

$$\log: \quad T \in SE(3) \quad \rightarrow \quad [\mathcal{S}]\theta \in se(3)$$

Screw axis $$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

Forward Kinematics



Forward kinematics of a 3R planar open chain.

- A different approach
- Define M to the position and orientation of frame {4} when all the joint angles are zeros ("home" or "zero" position of the robot)

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Consider each revolute joint as a zero-pitch screw-axis expressed in the {0} frame

For joint 3

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

Linear velocity of the origin of {0} in the {0} frame

$$v_3 = -\omega_3 \times q_3$$

$$q_3 = (L_1 + L_2, 0, 0)$$

Forward Kinematics

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = \theta_2 = 0)$$

$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = 0)$$

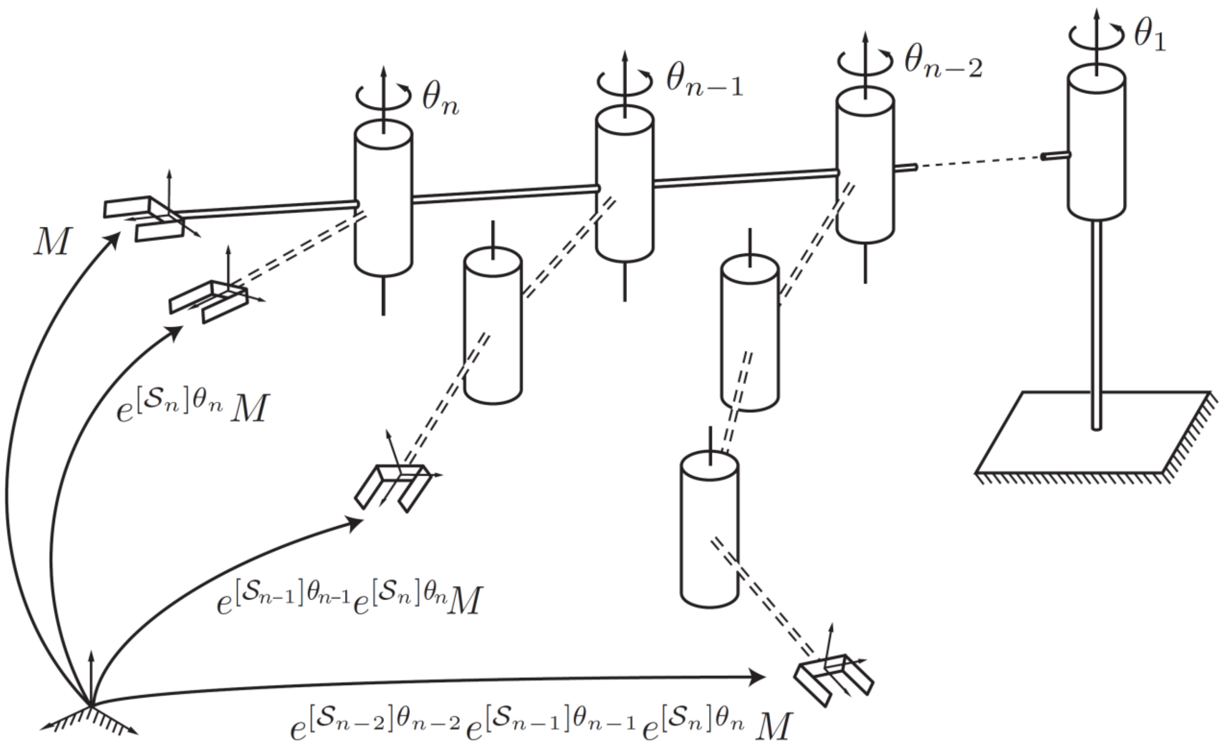
$$[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

a product of matrix exponentials
(does not use any frame references, only {0} and M)

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Product of Exponentials Formula



- Each link apply a screw motion to all the outward links
- Base frame {s}
- End-effector frame {b}

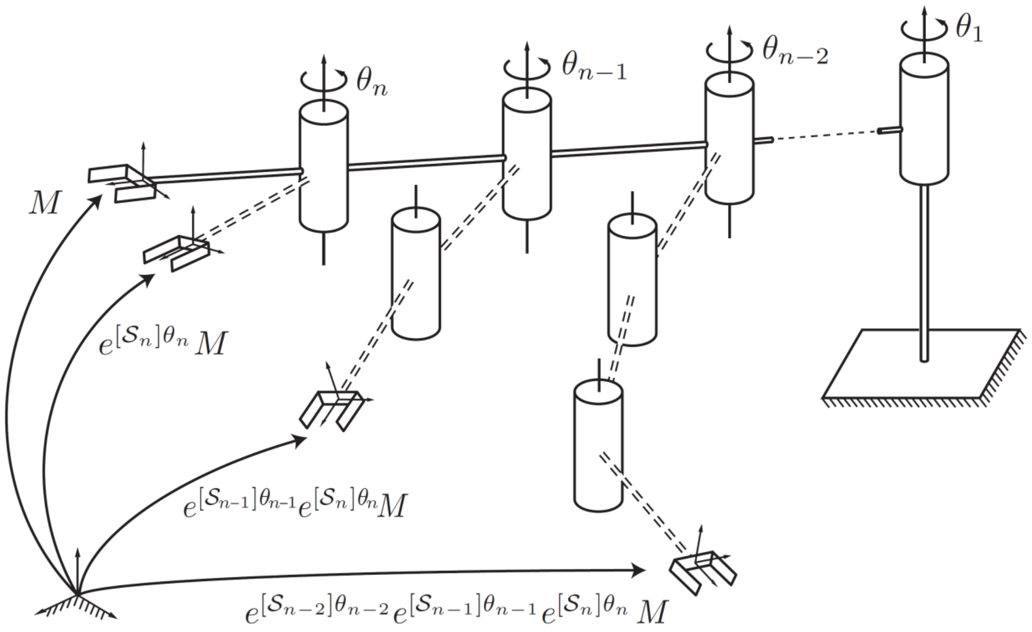
$$M \in SE(3)$$

{b} in {s} when all the joint values are zeros

$$T = e^{[\mathcal{S}_n]\theta_n} M$$

{b} in {s} when joint n with value θ_n

Product of Exponentials Formula

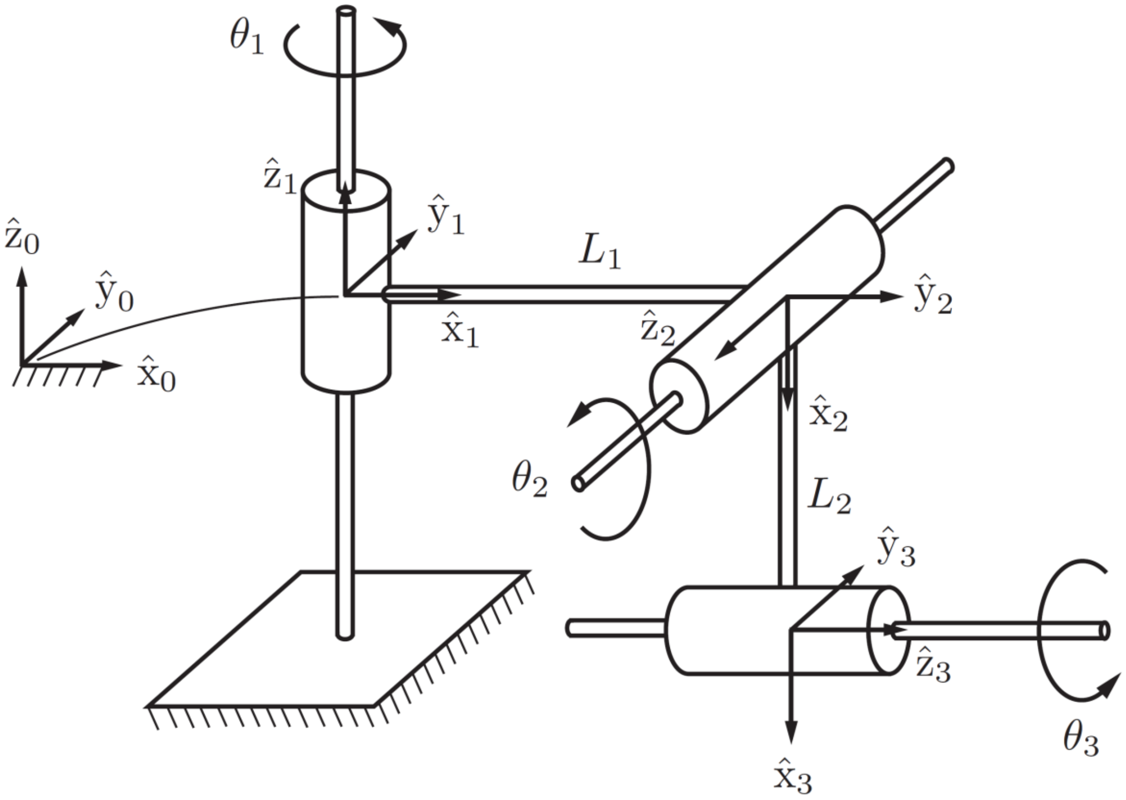


$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Joint values $(\theta_1, \ldots, \theta_n)$

- Space form of the product of exponentials formula
- Unlike D-H representation, no link reference frames need to be defined

Product of Exponentials Formula



A 3R spatial open chain

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{S}_1 = (\omega_1, v_1) \quad \omega_1 = (0,0,1) \quad v_1 = (0,0,0)$$

$$\omega_2 = (0,-1,0) \quad q_2 = (L_1, 0, 0)$$

$$v_2 = -\omega_2 \times q_2 = (0,0,-L_1)$$

$$\omega_3 = (1,0,0) \quad q_3 = (0,0,-L_2)$$

$$v_3 = -\omega_3 \times q_3 = (0,-L_2,0)$$

Cross Product

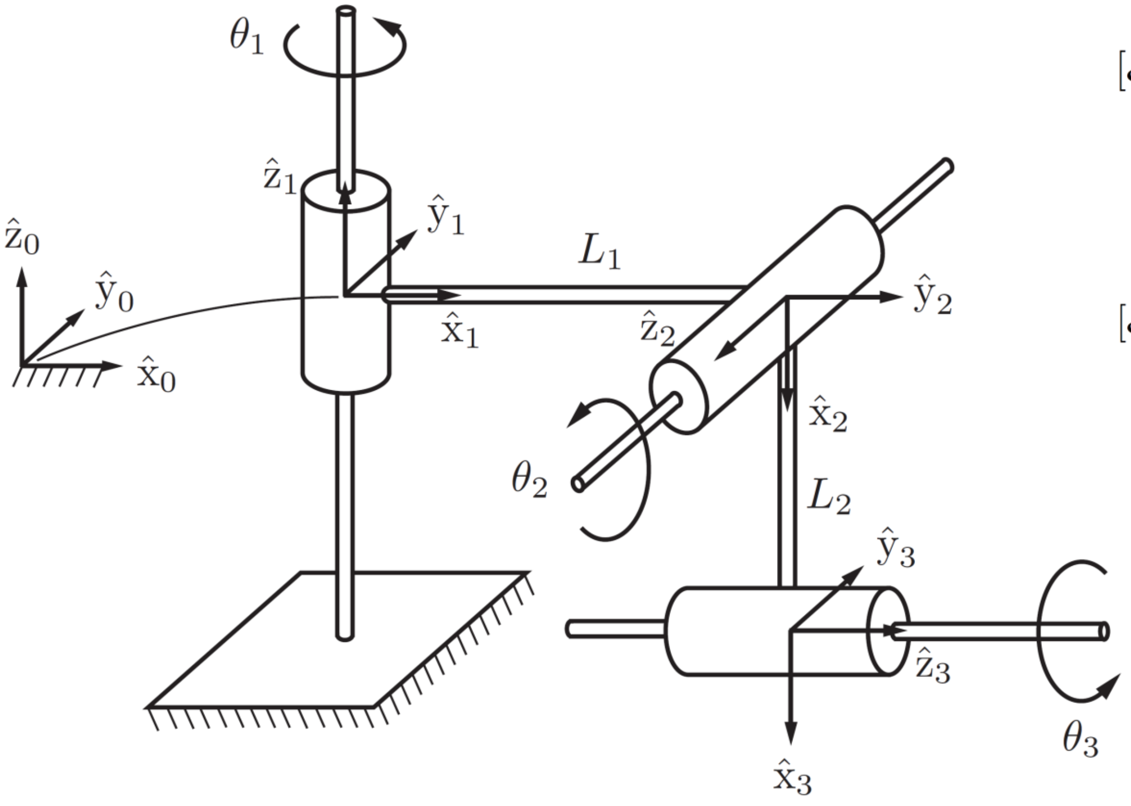
- Matrix notation

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\ &= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}, \end{aligned}$$

https://en.wikipedia.org/wiki/Cross_product

Product of Exponentials Formula



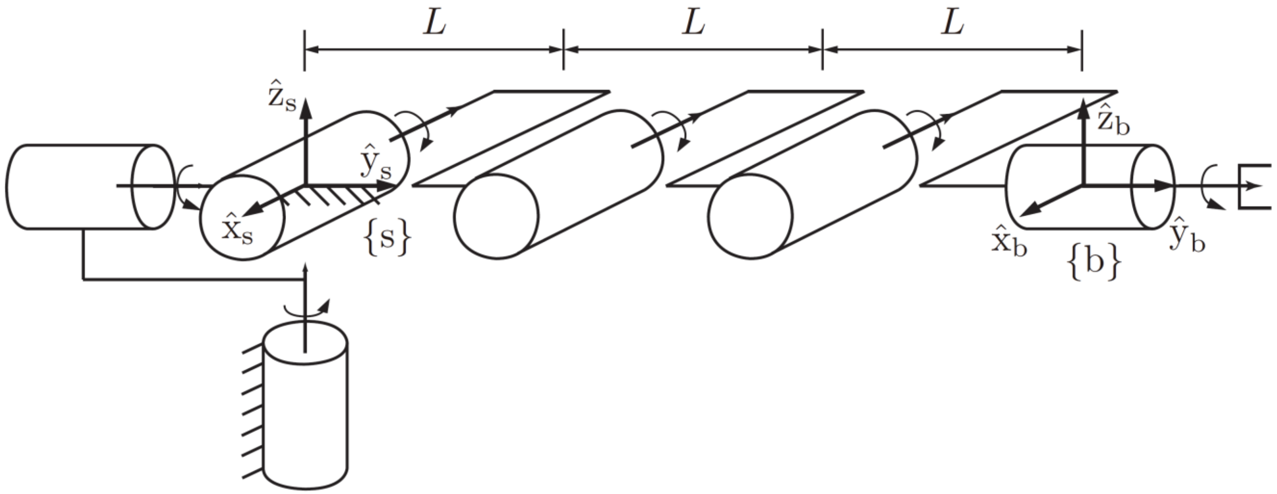
A 3R spatial open chain

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,-1,0)$	$(0,0,-L_1)$
3	$(1,0,0)$	$(0,L_2,0)$

Product of Exponentials Formula



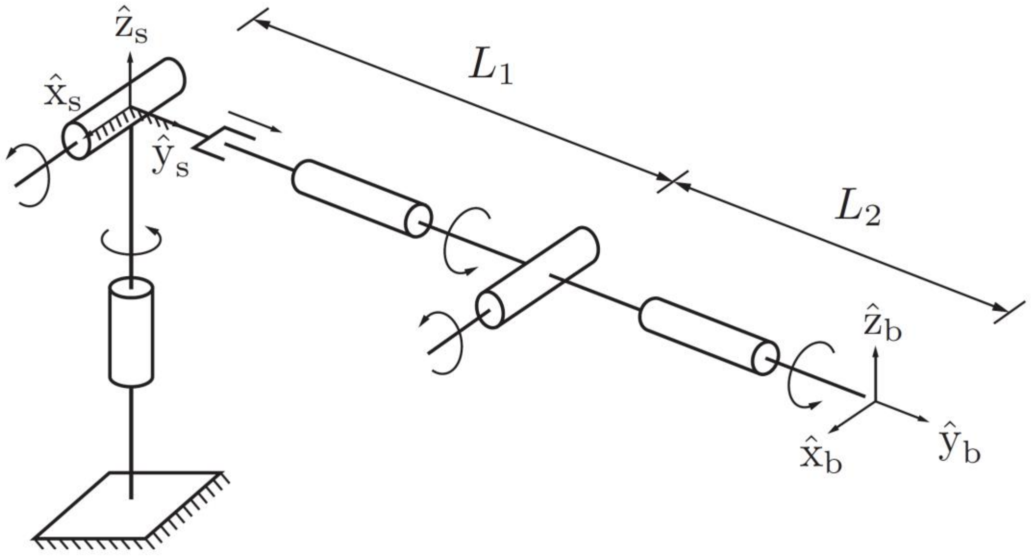
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, 1, 0)$	$(0, 0, 0)$
3	$(-1, 0, 0)$	$(0, 0, 0)$
4	$(-1, 0, 0)$	$(0, 0, L)$
5	$(-1, 0, 0)$	$(0, 0, 2L)$
6	$(0, 1, 0)$	$(0, 0, 0)$

PoE forward kinematics for the 6R open chain

First three joints are at the same location

Product of Exponentials Formula

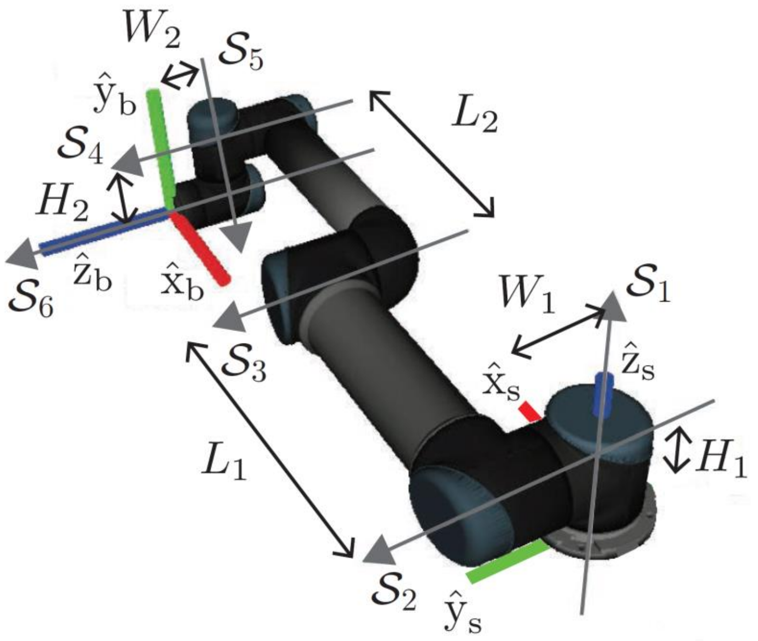


The RRPRRR spatial open chain

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(1,0,0)$	$(0,0,0)$
3	$(0,0,0)$	$(0,1,0)$
4	$(0,1,0)$	$(0,0,0)$
5	$(1,0,0)$	$(0,0,-L_1)$
6	$(0,1,0)$	$(0,0,0)$

Product of Exponentials Formula



Universal Robots' UR5 6R robot arm

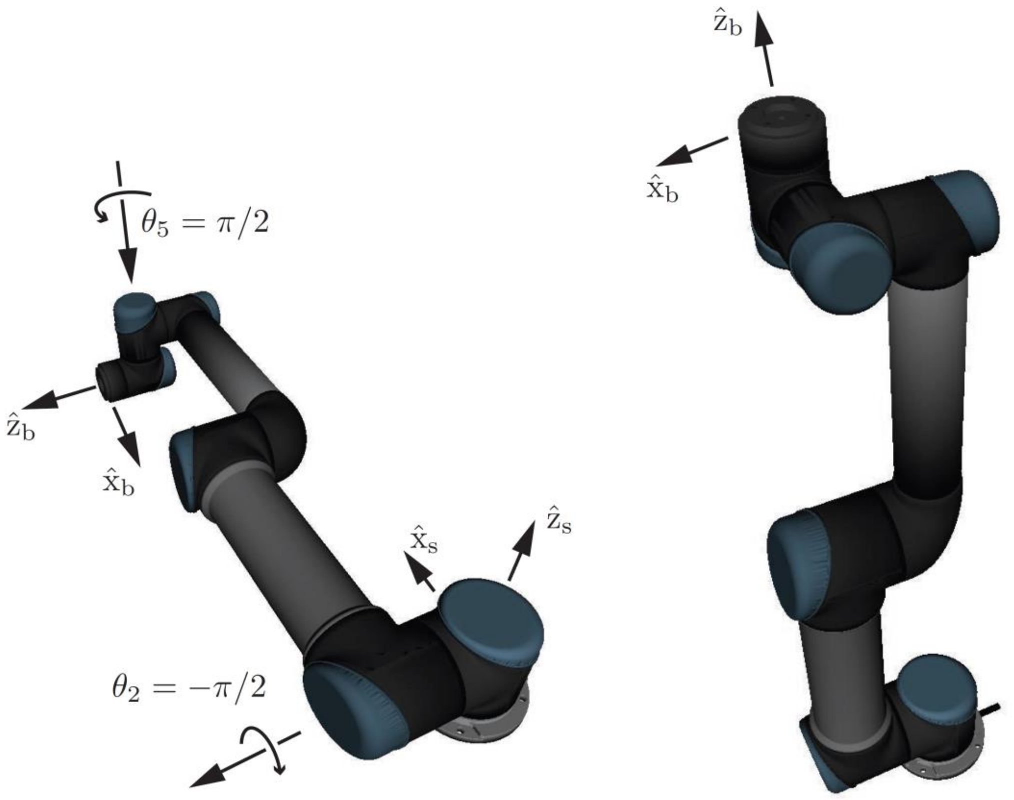
$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1,0,0)$
3	$(0,1,0)$	$(-H_1,0,L_1)$
4	$(0,1,0)$	$(-H_1,0,L_1+L_2)$
5	$(0,0,-1)$	$(-W_1,L_1+L_2,0)$
6	$(0,1,0)$	$(H_2-H_1,0,L_1+L_2)$

$W_1 = 109$ mm, $W_2 = 82$ mm, $L_1 = 425$ mm

$L_2 = 392$ mm, $H_1 = 89$ mm $H_2 = 95$ mm

Product of Exponentials Formula



Universal Robots' UR5 6R robot arm

$W_1 = 109$ mm, $W_2 = 82$ mm, $L_1 = 425$ mm

$L_2 = 392$ mm, $H_1 = 89$ mm $H_2 = 95$ mm

$$\theta_2 = -\pi/2 \text{ and } \theta_5 = \pi/2.$$

$$\begin{aligned} T(\theta) &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M \\ &= I e^{-[\mathcal{S}_2]\pi/2} I^2 e^{[\mathcal{S}_5]\pi/2} I M \\ &= e^{-[\mathcal{S}_2]\pi/2} e^{[\mathcal{S}_5]\pi/2} M \end{aligned}$$

$$e^{-[\mathcal{S}_2]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.089 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.089 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad e^{[\mathcal{S}_5]\pi/2} = \begin{bmatrix} 0 & 1 & 0 & 0.708 \\ -1 & 0 & 0 & 0.926 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T(\theta) = e^{-[\mathcal{S}_2]\pi/2} e^{[\mathcal{S}_5]\pi/2} M = \begin{bmatrix} 0 & -1 & 0 & 0.095 \\ 1 & 0 & 0 & 0.109 \\ 0 & 0 & 1 & 0.988 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall Twists

- Spatial twist (spatial velocity in the space frame)

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6 \qquad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3)$$

- Relationship

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}[\mathcal{V}_s]T \end{aligned} \qquad [\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}$$

Recall Adjoint Representations

- The adjoint representation of $T = (R, p) \in SE(3)$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- The adjoint map associated with T

$$\mathcal{V} \in \mathbb{R}^6 \qquad \mathcal{V}' = [\mathrm{Ad}_T]\mathcal{V} \quad \text{or} \quad \mathcal{V}' = \mathrm{Ad}_T(\mathcal{V})$$

$$[\mathcal{V}] \in se(3) \qquad [\mathcal{V}'] = T[\mathcal{V}]T^{-1}$$

Recall Twists

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s$$

In general

$$\mathcal{V}_c = [\mathrm{Ad}_{T_{cd}}]\mathcal{V}_d, \qquad \mathcal{V}_d = [\mathrm{Ad}_{T_{dc}}]\mathcal{V}_c$$

Screw Axes in the End-Effector Frame

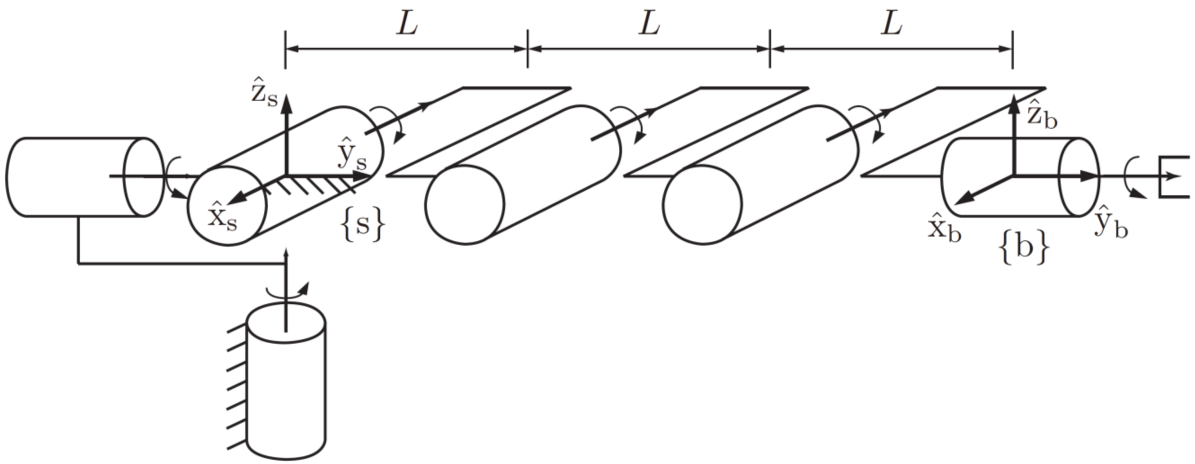
- Proposition $e^{M^{-1}PM} = M^{-1}e^{P}M$ $\quad Me^{M^{-1}PM} = e^{P}M$
- PoE formula

$$
\begin{aligned}
T(\theta) &= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= M e^{M^{-1}[\mathcal{S}_1]M\theta_1} \cdots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n},
\end{aligned}
$$

$[\mathcal{B}_i] = M^{-1}[\mathcal{S}_i]M$ $\qquad \mathcal{B}_i = [\mathrm{Ad}_{M^{-1}}]\mathcal{S}_i, \; i = 1, \ldots, n$

Body form of the product of exponentials formula

Screw Axes in the End-Effector Frame



PoE forward kinematics for the 6R open chain

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

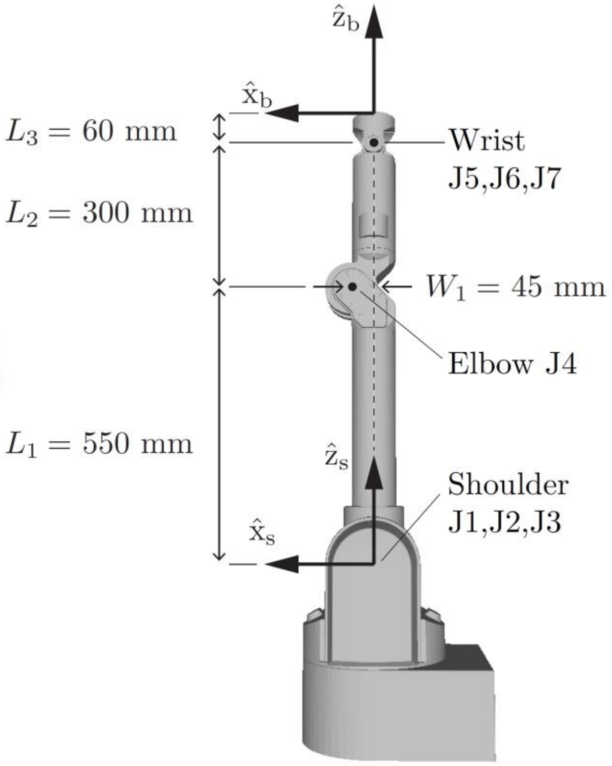
i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, 1, 0)$	$(0, 0, 0)$
3	$(-1, 0, 0)$	$(0, 0, 0)$
4	$(-1, 0, 0)$	$(0, 0, L)$
5	$(-1, 0, 0)$	$(0, 0, 2L)$
6	$(0, 1, 0)$	$(0, 0, 0)$

Space form

i	ω_i	v_i
1	$(0, 0, 1)$	$(-3L, 0, 0)$
2	$(0, 1, 0)$	$(0, 0, 0)$
3	$(-1, 0, 0)$	$(0, 0, -3L)$
4	$(-1, 0, 0)$	$(0, 0, -2L)$
5	$(-1, 0, 0)$	$(0, 0, -L)$
6	$(0, 1, 0)$	$(0, 0, 0)$

Body form

Screw Axes in the End-Effector Frame



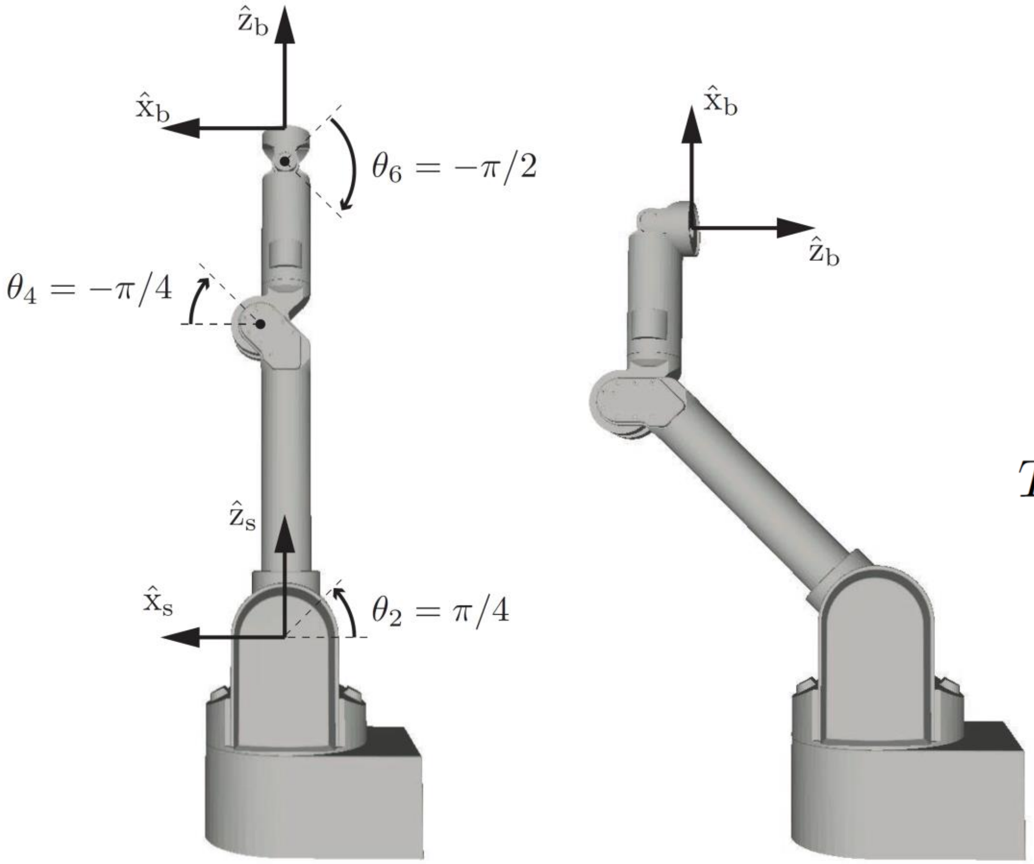
Barrett Technology's WAM 7R robot arm at its zero configuration

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{B}_i = (\omega_i, v_i)$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(L_1+L_2+L_3,0,0)$
3	$(0,0,1)$	$(0,0,0)$
4	$(0,1,0)$	$(L_2+L_3,0,W_1)$
5	$(0,0,1)$	$(0,0,0)$
6	$(0,1,0)$	$(L_3,0,0)$
7	$(0,0,1)$	$(0,0,0)$

Screw Axes in the End-Effector Frame



Barrett Technology's WAM 7R robot arm at its zero configuration

$$\theta_2 = 45°,\ \theta_4 = -45°,\ \theta_6 = -90°$$

$$T(\theta) = Me^{[\mathcal{B}_2]\pi/4}e^{-[\mathcal{B}_4]\pi/4}e^{-[\mathcal{B}_6]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.3157 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.6571 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary

- Forward kinematics
- Product of Exponentials Formula
  - Space form
  - Body form

Further Reading

- Chapter 4 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.