Screw Axes and Exponential Coordinates of Rigid-Body Motions

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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Twists

- Body twist (spatial velocity in the body frame)

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$$

matrix representation

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$$

$[\omega_b] \in so(3)$ $\quad v_b \in \mathbb{R}^3$ **linear velocity of a point at the origin of {b} expressed in {b}**

Twists

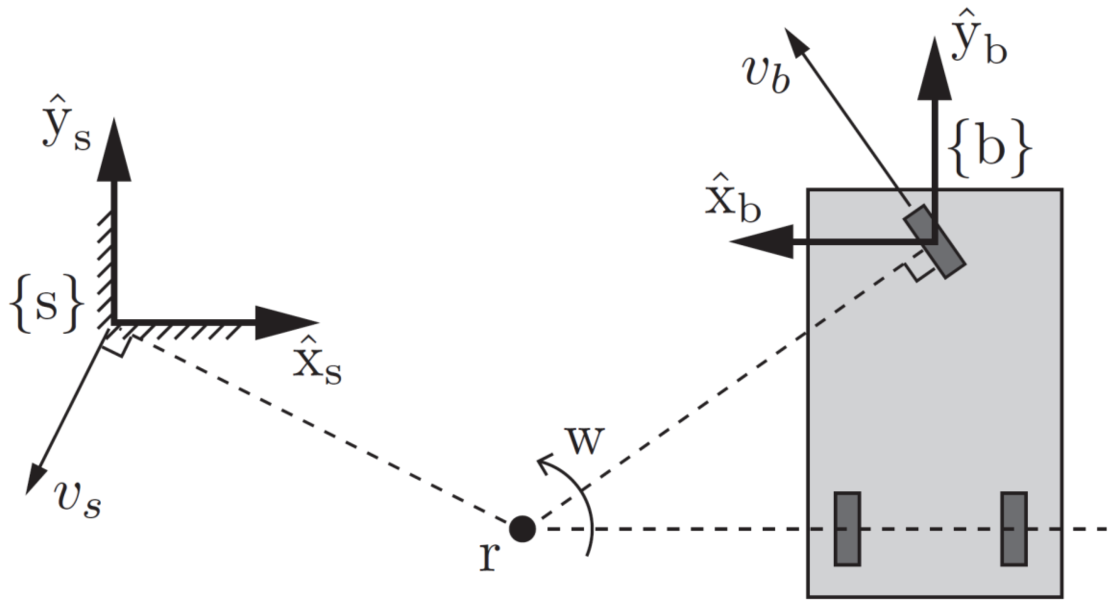
- Spatial twist (spatial velocity in the space frame)

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6 \qquad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3)$$

- Relationship

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}[\mathcal{V}_s]\,T \end{aligned} \qquad [\mathcal{V}_s] = T\,[\mathcal{V}_b]\,T^{-1}$$

Twists Example



Top view of a car

Rotation can cause linear velocity

- Pure Angular velocity $w = 2 \text{ rad/s}$

$r_s = (2, -1, 0)$ $r_b = (2, -1.4, 0)$

$\omega_s = (0, 0, 2)$ $\omega_b = (0, 0, -2)$

Linear velocity of the car

$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2, -4, 0),$$
$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8, 4, 0),$$

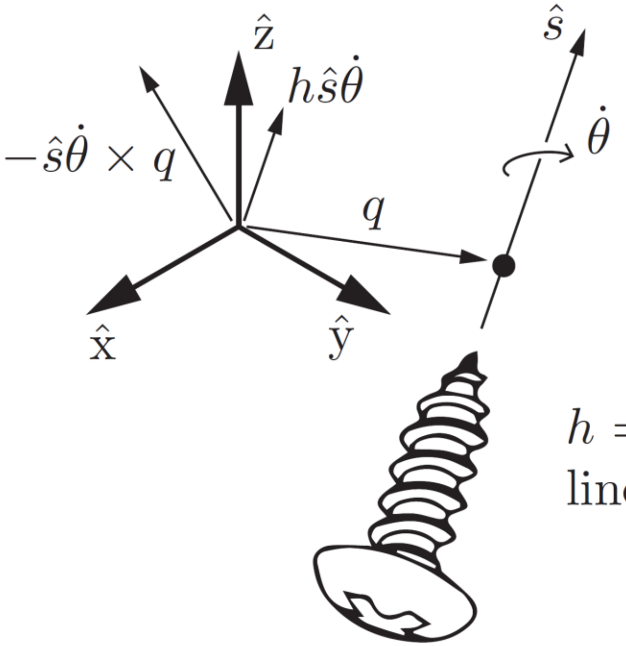
The Screw Interpretation of a Twist

- Screw axis: motion of a screw
  - Rotating about the axis while translating along the axis

https://mecharithm.com/learning/lesson/screw-motion-and-exponential-coordinates-of-robot-motions-11

The Screw Interpretation of a Twist

- Screw axis: motion of a screw
  - Rotating about the axis while translating along the axis



h = pitch = linear speed/angular speed

Screw axis $\mathcal{S}$ is the collection $\{q, \hat{s}, h\}$

$q \in \mathbb{R}^3$ is a point on the axis

Twist about S with angular velocity $\dot{\theta}$

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}$$

The Screw Interpretation of a Twist

- For any twist $\mathcal{V} = (\omega, v) \quad \omega \neq 0$
- These exists $\{q, \hat{s}, h\} \quad \dot{\theta}$

$$\hat{s} = \omega / \|\omega\| \qquad \dot{\theta} = \|\omega\| \qquad h = \hat{\omega}^{\mathrm{T}} v / \dot{\theta}$$

portion of v parallel to the screw axis

$-\hat{s}\dot{\theta} \times q$ provides the portion of v orthogonal to the screw axis (choose q based on this term)

If $\omega = 0$

$$\hat{s} = v / \|v\|$$

$\dot{\theta}$ is interpreted as the linear velocity $\|v\|$ along $\hat{s}$

The Screw Interpretation of a Twist

- Another representation of the screw axis

If $\omega \neq 0$ $\quad \mathcal{S} = \mathcal{V}/\|\omega\| = (\omega/\|\omega\|, v/\|\omega\|)$

$\mathcal{V} = (\omega, v)$ $\quad \dot{\theta} = \|\omega\| \quad \mathcal{S}\dot{\theta} = \mathcal{V}$

If $\omega = 0$ $\quad \mathcal{S} = \mathcal{V}/\|v\| = (0, v/\|v\|)$

$\dot{\theta} = \|v\| \quad \mathcal{S}\dot{\theta} = \mathcal{V}$

Screw Axis

- A screw axis is a normalized twist $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$ $\quad \mathcal{S}\dot{\theta} = \mathcal{V}$

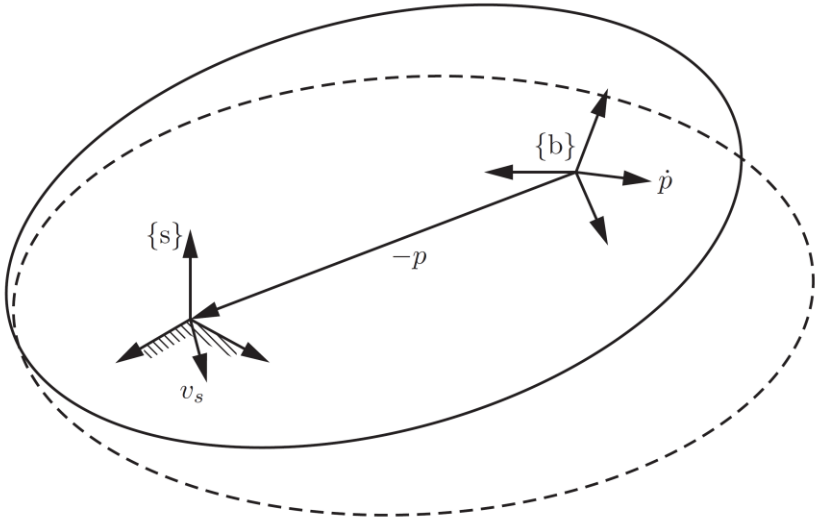
$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3) \qquad [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$

$$\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \qquad \mathcal{S}_b = [\mathrm{Ad}_{T_{ba}}]\mathcal{S}_a$$

Twists and Screw Axes

- Twist

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6 \quad \mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6$$



- A screw axis is a normalized twist

$$\mathcal{S}\dot{\theta} = \mathcal{V} \qquad \mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

Exponential Coordinates of Rigid-Body Motions

- **Chasles-Mozzi theorem**: every rigid-body displacement can be expressed as displacement along a fixed screw axis S in space

- Exponential coordinates of a homogeneous transformation T

$$\mathcal{S}\theta \in \mathbb{R}^6$$

Screw axis — Distance along the screw axis

$$\mathcal{S} = (\omega, v) \qquad \|\omega\| = 1 \qquad \theta \text{ Angle of rotation}$$

$$\omega = 0 \quad \|v\| = 1 \qquad \theta \text{ Linear distance along the axis}$$

Exponential Coordinates of Rigid-Body Motions

- Exponential coordinates of a homogeneous transformation T

$$\exp: \quad [\mathcal{S}]\theta \in se(3) \quad \rightarrow \quad T \in SE(3)$$

$$\log: \quad T \in SE(3) \quad \rightarrow \quad [\mathcal{S}]\theta \in se(3)$$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

Matrix Exponential

$$e^{[\mathcal{S}]\theta} = I + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \cdots$$

$$= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

$$G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \cdots$$

$$[\omega]^3 = -[\omega]$$

$$= I\theta + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\omega] + \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \cdots\right)[\omega]^2$$

$$= I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2$$

Matrix Exponential

$$\mathcal{S} = (\omega, v) \qquad \theta \in \mathbb{R}$$

If $\|\omega\| = 1$ $\quad e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}$

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \in SO(3)$$

If $\omega = 0$ and $\|v\| = 1$ $\quad e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$

Matrix Logarithm

- Given $(R, p) \in SE(3)$, one can find $\mathcal{S} = (\omega, v)$ and θ

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

- Matrix Logarithm of T = (R, p)

$$[\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix} \in se(3)$$

Matrix Logarithm Algorithm

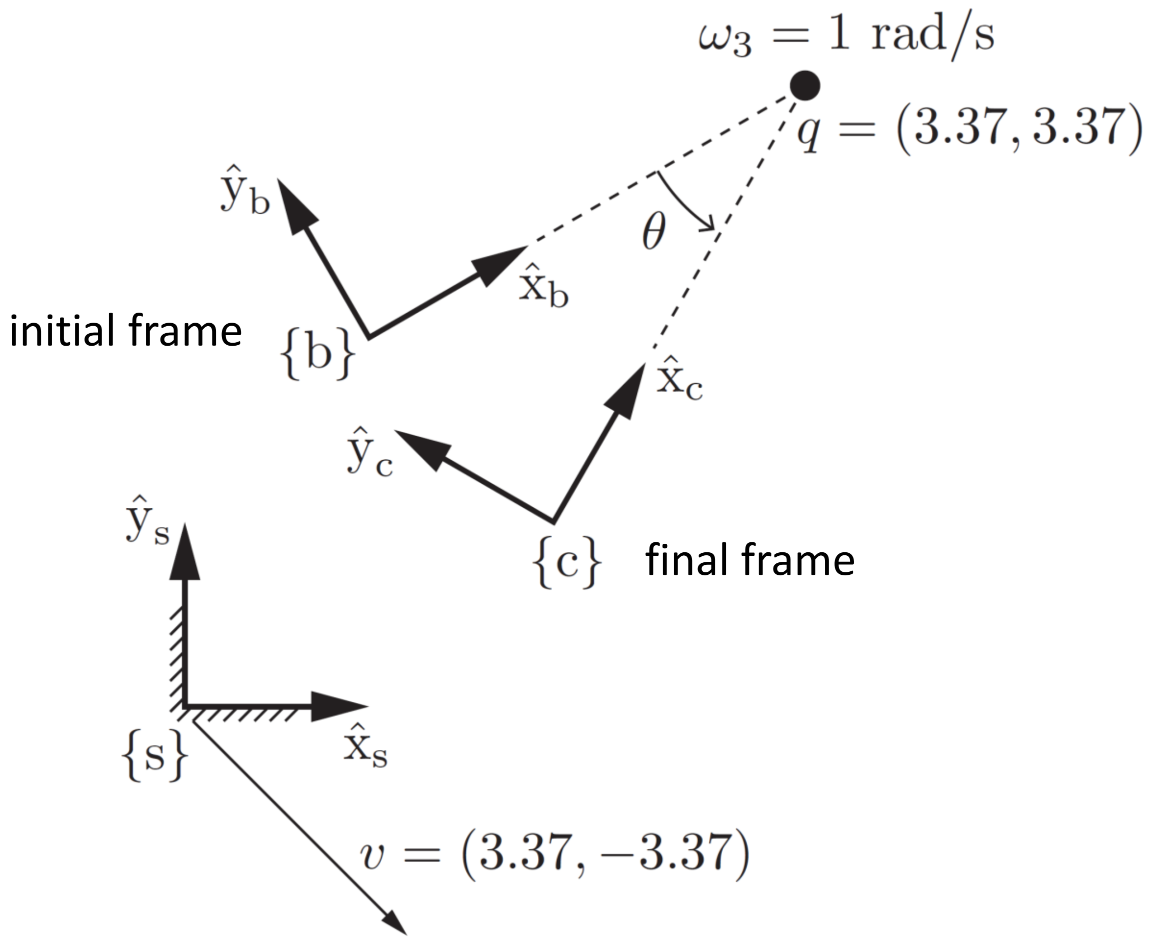
- Given $(R, p) \in SE(3)$, how to find $\mathcal{S} = (\omega, v)$ and θ ?
  - If $R = I$ then set $\omega = 0,\ v = p/\|p\|$, and $\theta = \|p\|$
  - Otherwise, use the matrix logarithm on SO(3) to determine ω, θ for R (lecture 7)

$$v = G^{-1}(\theta)p$$

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2$$

Exercise

Matrix Exponential and Matrix Logarithm



$$T_{sb} = \begin{bmatrix} \cos 30° & -\sin 30° & 0 & 1 \\ \sin 30° & \cos 30° & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{sc} = \begin{bmatrix} \cos 60° & -\sin 60° & 0 & 2 \\ \sin 60° & \cos 60° & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

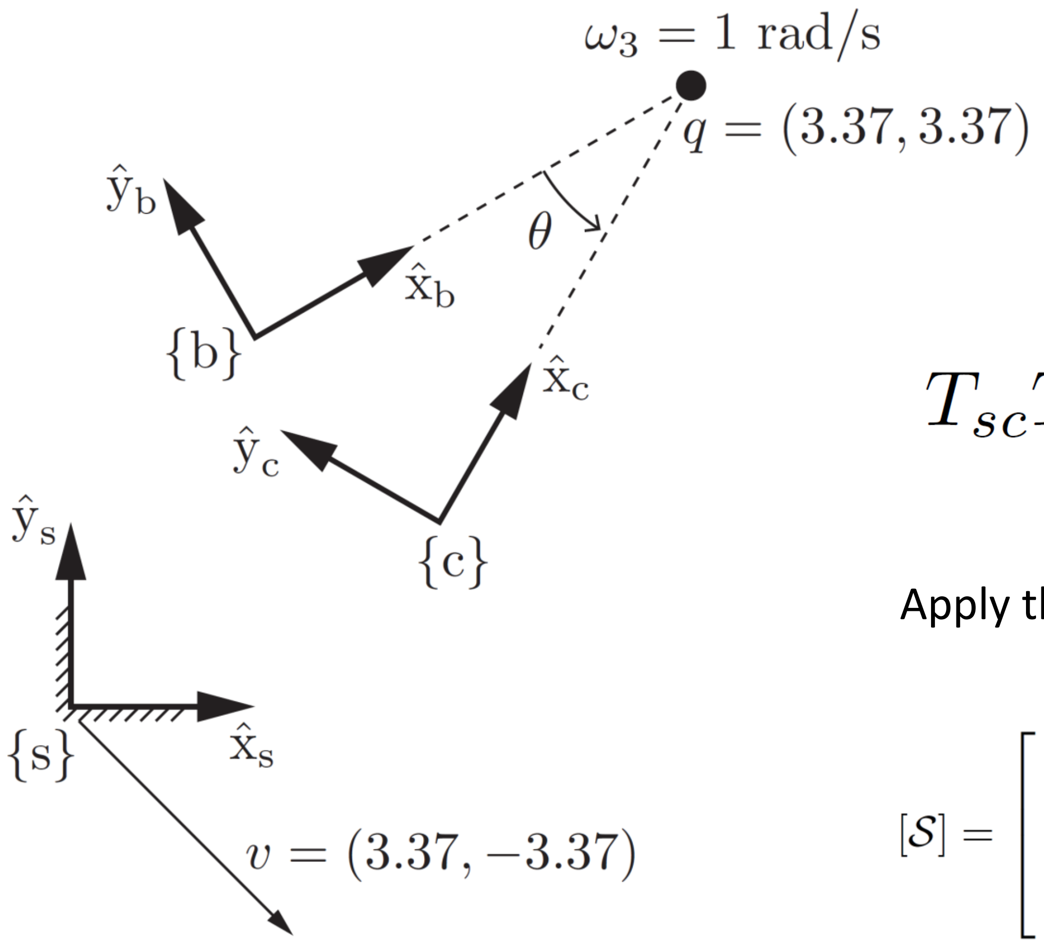
motion occurs in the $\hat{x}_s$–$\hat{y}_s$-plane

Screw axis: $\hat{z}_s$-axis   Zero pitch

$$\mathcal{S} = (\omega, v) \qquad \begin{aligned} \omega &= (0, 0, \omega_3), \\ v &= (v_1, v_2, 0). \end{aligned}$$

In {s}

Matrix Exponential and Matrix Logarithm



Seek screw motion to displace {b} to {c}

$$T_{sc} = e^{[\mathcal{S}]\theta} T_{sb}$$

$$T_{sc}T_{sb}^{-1} = e^{[\mathcal{S}]\theta} \qquad [\mathcal{S}] = \begin{bmatrix} 0 & -\omega_3 & 0 & v_1 \\ \omega_3 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply the matrix logarithm algorithm to $T_{sc}T_{sb}^{-1}$

$$[\mathcal{S}] = \begin{bmatrix} 0 & -1 & 0 & 3.37 \\ 1 & 0 & 0 & -3.37 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3.37 \\ -3.37 \\ 0 \end{bmatrix}, \quad \theta = \frac{\pi}{6} \text{ rad (or } 30^\circ)$$

Summary

- Screw Axes
- Exponential Coordinates of Rigid-Body Motions
- Matrix Logarithm of Rigid-Body Motions

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017
- Basics of Classical Lie Groups: The Exponential Map, Lie Groups, and Lie Algebras https://www.cis.upenn.edu/~cis6100/geombchap14.pdf
- Exponential Coordinate of Rigid Body Configuration. Prof. Wei Zhang, Southern University of Science and Technology, Shenzhen, China http://www.wzhanglab.site/wp-content/uploads/2021/06/LN4_ExpCoordinate-a-print.pdf