Matrix Logarithm of Rotations and Homogeneous Transformation Matrices

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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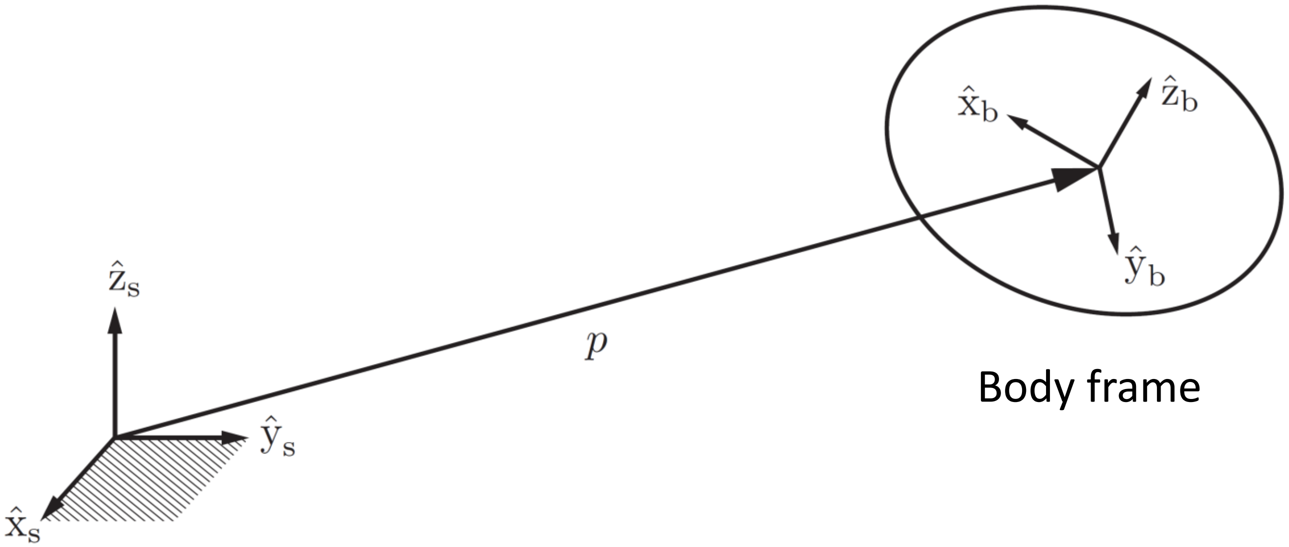
Rigid-Body in 3D



Fixed frame

Origin of the body frame

$$p = p_1 \hat{x}_s + p_2 \hat{y}_s + p_3 \hat{z}_s$$

Axes of the body frame

$$\begin{aligned} \hat{x}_b &= r_{11}\hat{x}_s + r_{21}\hat{y}_s + r_{31}\hat{z}_s, \\ \hat{y}_b &= r_{12}\hat{x}_s + r_{22}\hat{y}_s + r_{32}\hat{z}_s, \\ \hat{z}_b &= r_{13}\hat{x}_s + r_{23}\hat{y}_s + r_{33}\hat{z}_s. \end{aligned}$$

Translation

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$R = [\hat{x}_b \;\; \hat{y}_b \;\; \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

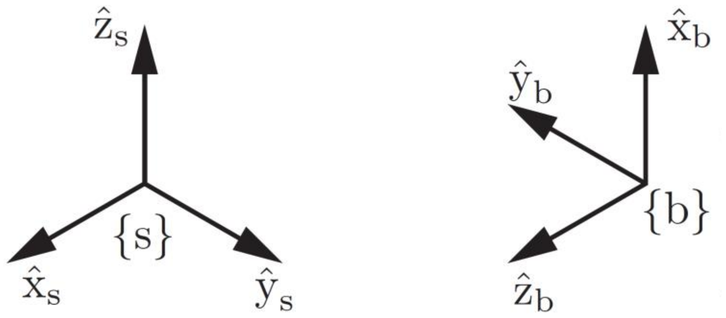
Rotation matrix

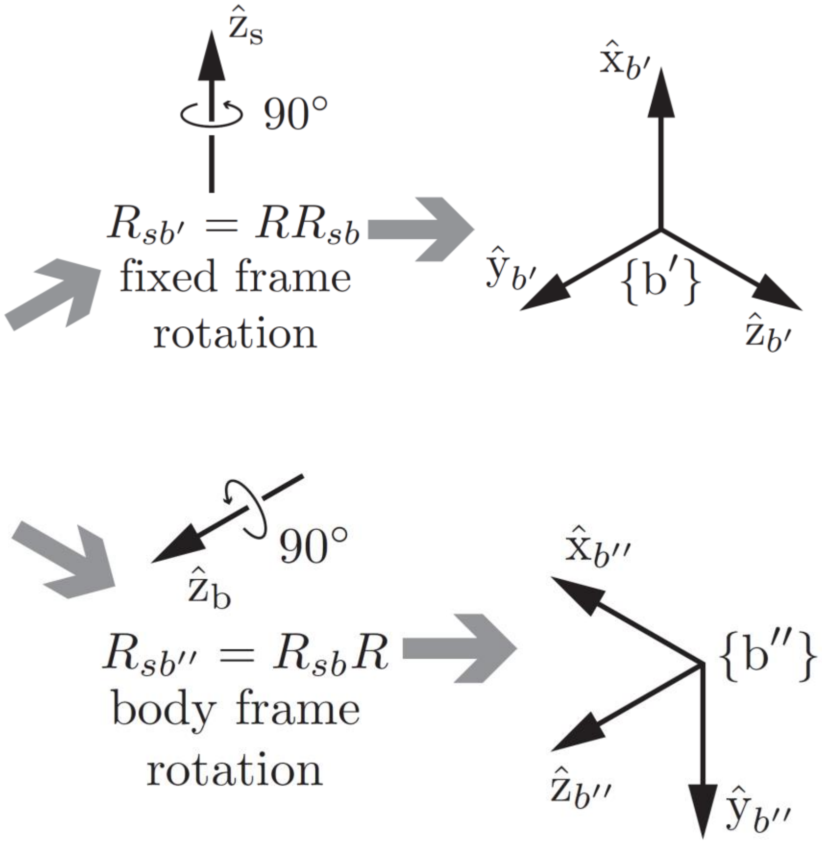
Rotating a Vector or a Frame

- {b} in {s} R_{sb}
- Rotate {b} with

$$\mathrm{Rot}(\hat{\omega}, \theta)$$

$\hat{\omega}$ represented in {s} or {b}?





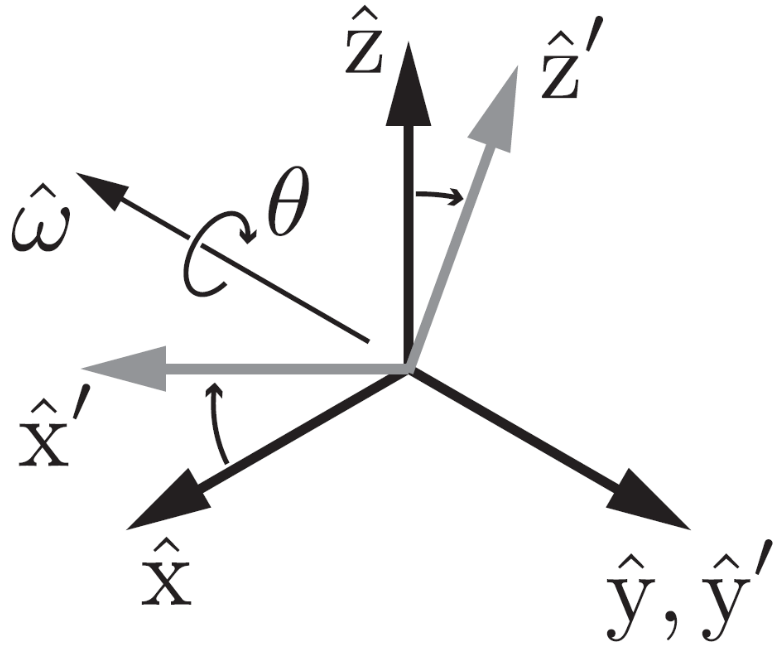
$$R_{sb'} = \text{rotate\_by\_}R\text{\_in\_\{s\}\_frame}\ (R_{sb}) = RR_{sb}$$

$$R_{sb''} = \text{rotate\_by\_}R\text{\_in\_\{b\}\_frame}\ (R_{sb}) = R_{sb}R$$

Exponential Coordinates of Rotations

- Exponential coordinates
  - A rotation axis (unit length) $\hat{\omega}$
  - An angle of rotation about the axis θ

$$\hat{\omega}\theta \in \mathbb{R}^3$$



Fix the origin when rotating

Matrix Logarithm of Rotations

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Skew-symmetric Matrix

- Rodrigues' formula

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta \, [\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

- If $\hat{\omega}\theta \in \mathbb{R}^3$ represent the exponential coordinates of rotation R, then the matrix logarithm of the rotation R is

$$[\hat{\omega}\theta] = [\hat{\omega}]\theta$$

$$\begin{aligned} \exp : \quad & [\hat{\omega}]\theta \in so(3) & \rightarrow \quad & R \in SO(3), \\ \log : \quad & R \in SO(3) & \rightarrow \quad & [\hat{\omega}]\theta \in so(3). \end{aligned}$$

Matrix Logarithm of Rotations

How to compute matrix logarithm?

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \in SO(3)$$

$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3) \quad s_\theta = \sin\theta \quad c_\theta = \cos\theta$$

$$\begin{aligned} r_{32} - r_{23} &= 2\hat{\omega}_1 \sin\theta, \\ r_{13} - r_{31} &= 2\hat{\omega}_2 \sin\theta, \\ r_{21} - r_{12} &= 2\hat{\omega}_3 \sin\theta. \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \hat{\omega}_1 &= \frac{1}{2\sin\theta}(r_{32} - r_{23}), \\ \hat{\omega}_2 &= \frac{1}{2\sin\theta}(r_{13} - r_{31}), \\ \hat{\omega}_3 &= \frac{1}{2\sin\theta}(r_{21} - r_{12}). \end{aligned}$$

Matrix Logarithm of Rotations

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} = \frac{1}{2\sin\theta}\left(R - R^{\mathrm{T}}\right) \qquad \sin\theta \neq 0$$

$$\mathrm{tr}\, R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta \qquad \hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$$

When $\theta = k\pi$

- Even k, R=I, $\hat{\omega}$ undefined
- Odd k, $\theta = \pm\pi, \pm 3\pi, \ldots,$ $R = e^{[\hat{\omega}]\pi} = I + 2[\hat{\omega}]^2$ $\mathrm{tr}\, R = -1$

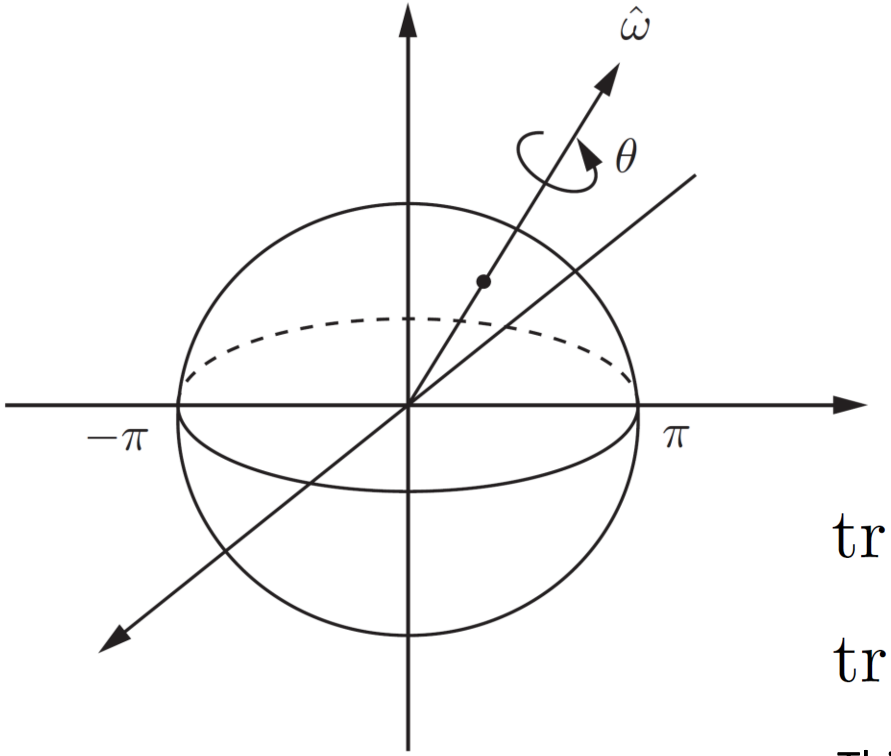
Solve this equation to compute $\hat{\omega}$

Matrix Logarithm of Rotations

- Solutions exist for $\theta \in [0, 2\pi]$
- We can restrict the solution to $\theta \in [0, \pi]$ Why?
- See algorithm in Page 74 in Lynch & Park

Exponential Coordinates and Matrix Logarithm

- Since exponential coordinates $\hat{\omega}\theta$ satisfies $||\hat{\omega}\theta|| \leq \pi$ $\quad \theta \in [0, \pi]$



Rotation axis can take negative direction

$$r \in \mathbb{R}^3 \text{ in this solid ball}$$

$$\hat{\omega} = r/\|r\|$$

$$\theta = \|r\| \qquad r = \hat{\omega}\theta$$

$$\operatorname{tr} R \neq -1 \quad e^{[r]} = R$$

$$\operatorname{tr} R = -1 \quad R = e^{[r]} \text{ with } \|r\| = \pi \qquad R = e^{[-r]}$$

This is because $R = e^{[\hat{\omega}]\pi} = I + 2[\hat{\omega}]^2$ and $[\hat{\omega}]^2 = [-\hat{\omega}]^2$

Exponential Coordinates of Rotations

$$\begin{aligned}\exp: &\quad [\hat{\omega}]\theta \in so(3) &\rightarrow&\quad R \in SO(3),\\ \log: &\quad R \in SO(3) &\rightarrow&\quad [\hat{\omega}]\theta \in so(3).\end{aligned}$$

Homogeneous Transformation Matrices

- Consider body frame {b} in a fixed frame {s}
  - 3D rotation $R \in SO(3)$
  - 3D position $p \in \mathbb{R}^3$

- Special Euclidean group SE(3) or homogenous transformation matrices

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation Matrices

- For planar motions, we have SE(2)

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \qquad R \in SO(2) \qquad p \in \mathbb{R}^2$$

$$T = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \theta \in [0, 2\pi)$$

Properties of Transformation Matrices

- The inverse of a transformation matrix is also a transformation matrix

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$$

- Closure $T_1 T_2$
- Associativity $(T_1 T_2) T_3 = T_1 (T_2 T_3)$
- Identity element: identity matrix I
- Not commutative $T_1 T_2 \neq T_2 T_1$

Homogeneous Coordinates

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = w \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene coordinates

Up to scale

Conversion

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Homogeneous Coordinates

$$T\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$$

Homogeneous transformation

Homogeneous coordinates

Properties of Transformation Matrices

Proposition 3.18. *Given* $T = (R, p) \in SE(3)$ *and* $x, y \in \mathbb{R}^3$, *the following hold:*

(a) $\|Tx - Ty\| = \|x - y\|$, *where* $\|\cdot\|$ *denotes the standard Euclidean norm in* $\mathbb{R}^3$, *i.e.*, $\|x\| = \sqrt{x^{\mathrm{T}} x}$.

Reserve distances

(b) $\langle Tx - Tz, Ty - Tz \rangle = \langle x - z, y - z \rangle$ *for all* $z \in \mathbb{R}^3$, *where* $\langle \cdot, \cdot \rangle$ *denotes the standard Euclidean inner product in* $\mathbb{R}^3$, $\langle x, y \rangle = x^{\mathrm{T}} y$.

Reserve angles

SE(3) can be identified with rigid-body motions

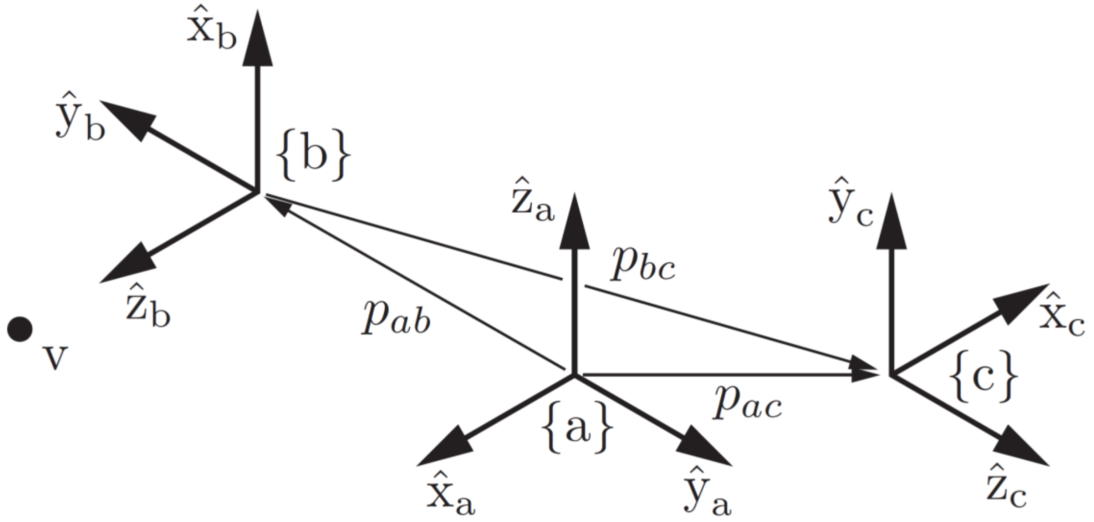
Uses of Transformation Matrices

- Represent the configuration of a rigid-body

- Change the reference frame

- Displace a vector or a frame

Representing a Configuration



$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$T_{sa} = (R_{sa}, p_{sa}) \quad T_{sb} = (R_{sb}, p_{sb})$$

$$T_{sc} = (R_{sc}, p_{sc})$$

$$T_{bc} = (R_{bc}, p_{bc}) \quad R_{bc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$p_{bc} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix} \qquad T_{de} = T_{ed}^{-1}$$

Changing the Reference Frame

$$T_{ab}T_{bc} = T_{a\not{b}}T_{\not{b}c} = T_{ac}$$

$$T_{ab}v_b = T_{a\not{b}}v_{\not{b}} = v_a$$

Displacing a Vector or a Frame

- Rotating and then translating $(R, p) = (\mathrm{Rot}(\hat{\omega}, \theta), p)$
- Transformation matrices

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathrm{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

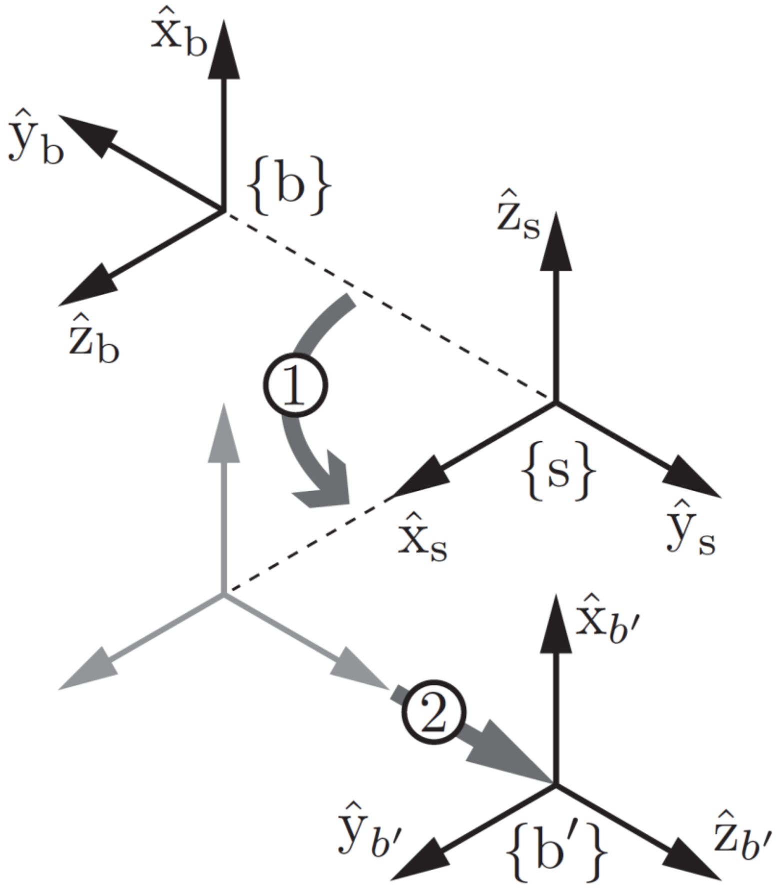
$$T_{sb'} = TT_{sb} = \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega}, \theta)T_{sb} \qquad \text{(fixed frame)}$$

$$= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & Rp_{sb} + p \\ 0 & 1 \end{bmatrix}$$

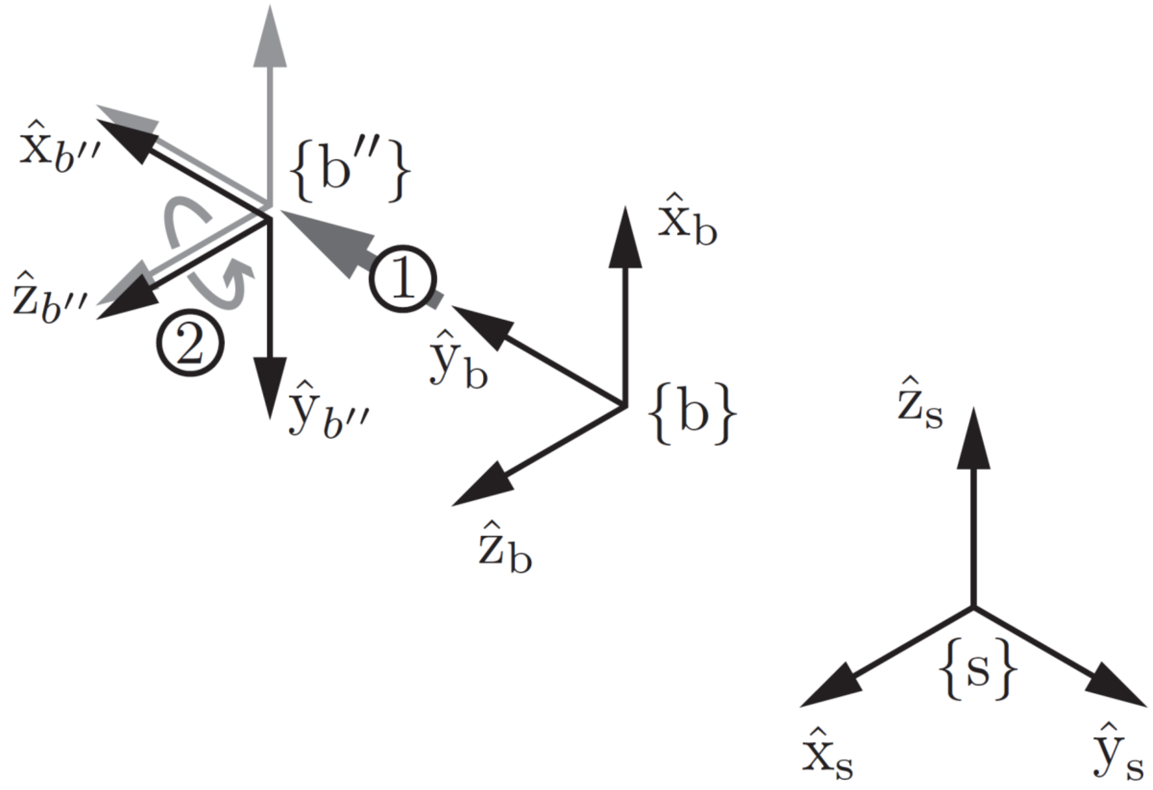
$$T_{sb''} = T_{sb}T = T_{sb}\,\mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega}, \theta) \qquad \text{(body frame)}$$

$$= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \\ 0 & 1 \end{bmatrix}$$

Displacing a Vector or a Frame



Fixed Frame



Body Frame

$$\hat{\omega} = (0, 0, 1) \quad \theta = 90^\circ \quad p = (0, 2, 0)$$

Summary

- Matrix Logarithm of Rotations

- Homogenous transformation matrices

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017