Angular Velocities and Exponential Coordinates of Rotations

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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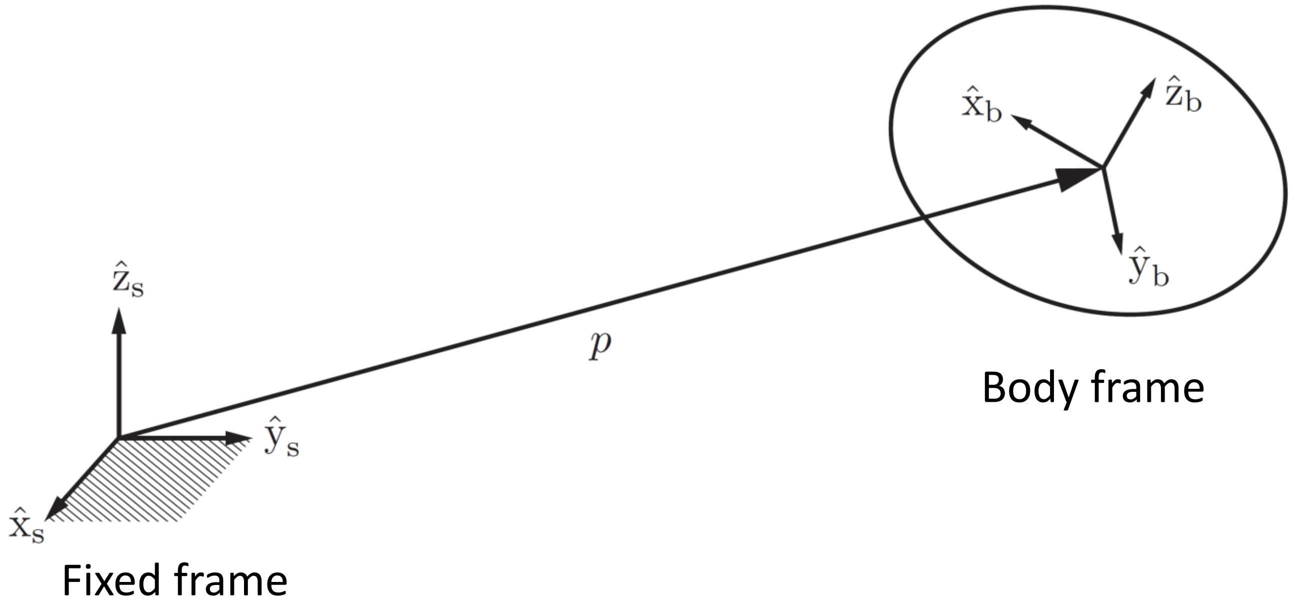
Rigid-Body in 3D



- Origin of the body frame

$$p = p_1 \hat{x}_s + p_2 \hat{y}_s + p_3 \hat{z}_s$$

- Axes of the body frame

$$\begin{aligned} \hat{x}_b &= r_{11}\hat{x}_s + r_{21}\hat{y}_s + r_{31}\hat{z}_s, \\ \hat{y}_b &= r_{12}\hat{x}_s + r_{22}\hat{y}_s + r_{32}\hat{z}_s, \\ \hat{z}_b &= r_{13}\hat{x}_s + r_{23}\hat{y}_s + r_{33}\hat{z}_s. \end{aligned}$$

Translation $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

Rotation matrix

$$R = [\hat{x}_b \;\; \hat{y}_b \;\; \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Meanings of the column vectors

Rotating a Vector or a Frame

- Rotate frame {c} about a unit axis $\hat{\omega}$ by θ to get frame {c'}, {c} is aligned with {s} in the beginning

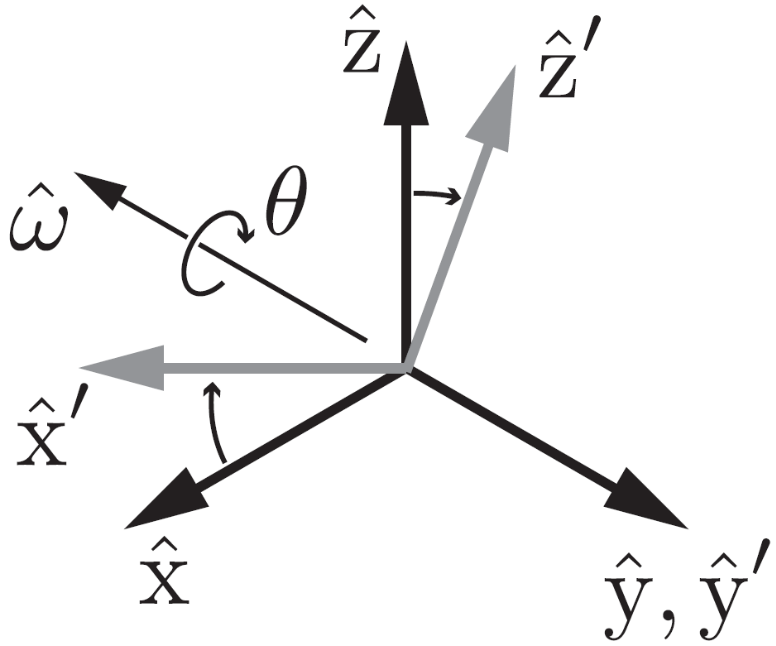
$$R = R_{sc'}$$

frame {c'} relative to frame {s}

- Rotation operation

$$R = \mathrm{Rot}(\hat{\omega}, \theta)$$

To rotate a vector $v' = Rv$

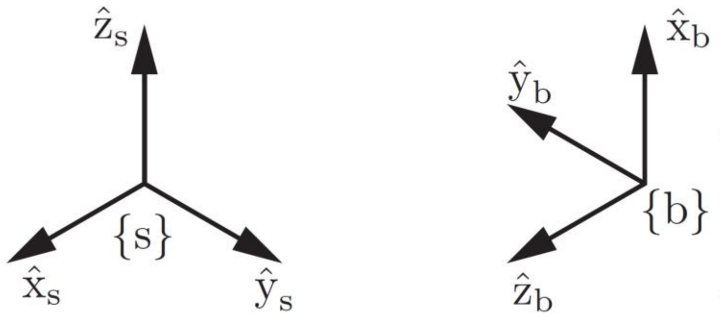


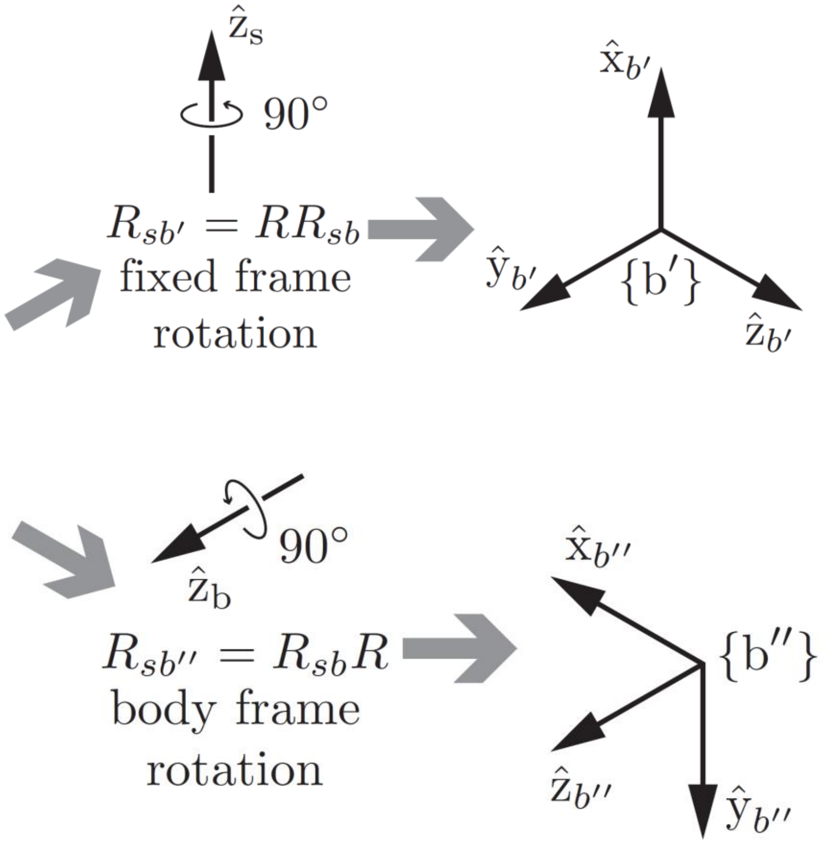
Rotating a Vector or a Frame

- {b} in {s} R_{sb}
- Rotate {b} with

$$\mathrm{Rot}(\hat{\omega}, \theta)$$

$\hat{\omega}$ represented in {s} or {b}?

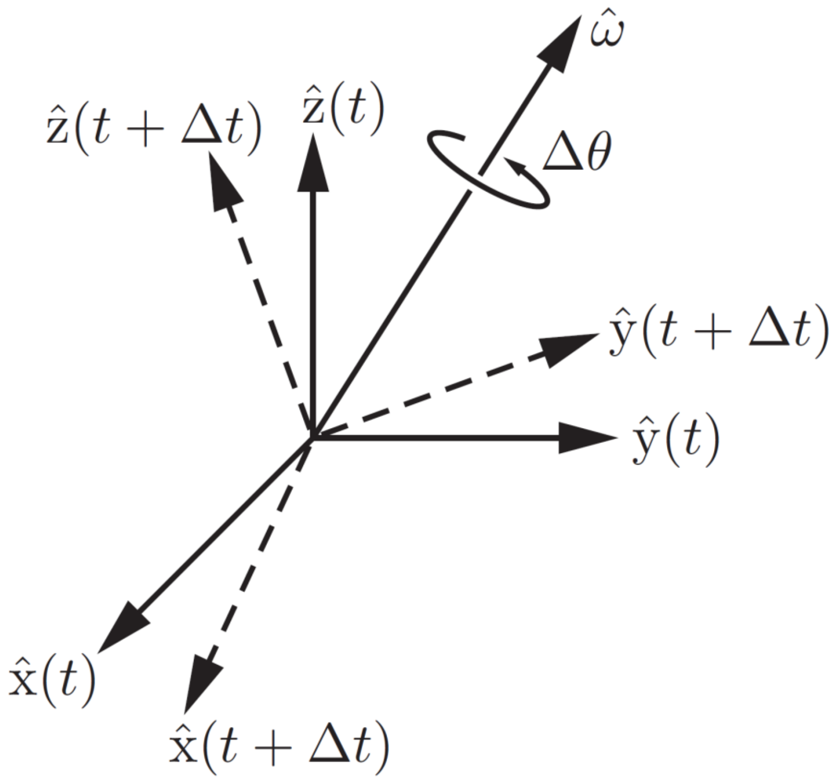




$$R_{sb'} = \text{rotate\_by\_}R\text{\_in\_\{s\}\_frame } (R_{sb}) = RR_{sb}$$

$$R_{sb''} = \text{rotate\_by\_}R\text{\_in\_\{b\}\_frame } (R_{sb}) = R_{sb}R$$

Angular Velocities



- Axes $\{\hat{x}, \hat{y}, \hat{z}\}$ Unit length
- Compute time derivates of these axes caused by rotation $\dot{\hat{x}}$

Rotating around $\hat{w}$ by $\Delta\theta$

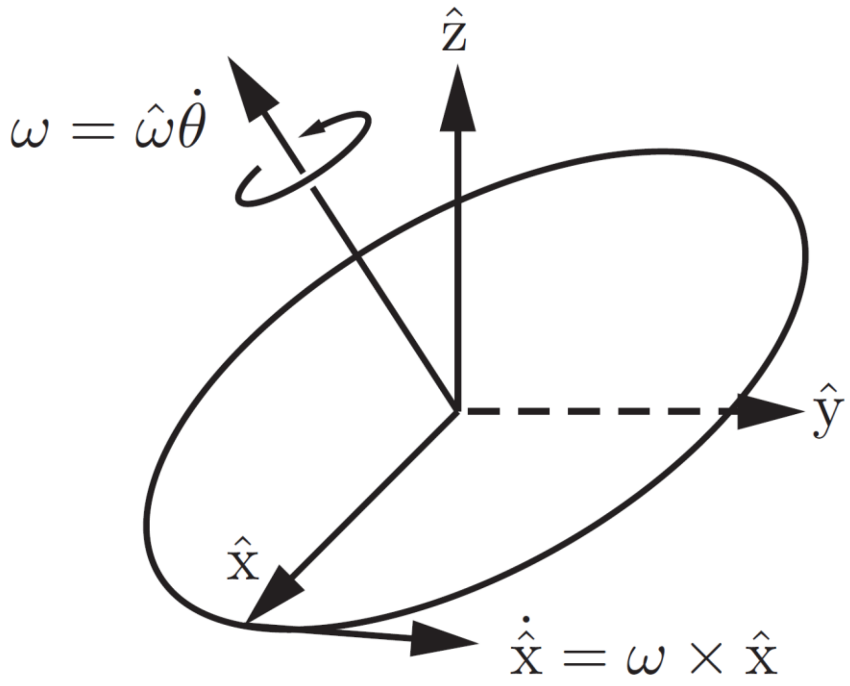
$\hat{w}$ is coordinate free for now

$\Delta t \to 0 \qquad \Delta\theta/\Delta t \longrightarrow \dot{\theta}$

$\hat{w}$ instantaneous axis of rotation

- Angular velocity $\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}$

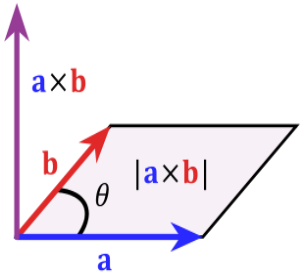
Angular Velocities



- Angular velocity $\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}$
- Compute time derivates of these axes caused by rotation $\dot{\hat{\mathrm{x}}}$

$$\begin{aligned} \dot{\hat{\mathrm{x}}} &= \mathrm{w} \times \hat{\mathrm{x}}, \\ \dot{\hat{\mathrm{y}}} &= \mathrm{w} \times \hat{\mathrm{y}}, \\ \dot{\hat{\mathrm{z}}} &= \mathrm{w} \times \hat{\mathrm{z}}. \end{aligned}$$

Vector cross product



https://en.wikipedia.org/wiki/Cross_product

Angular Velocities

- To express these equations in coordinates, we have to choose a reference frame for w
  - Two natural choices: fixed frame {s} or body frame {b}

- Consider fixed frame {s}
  - Orientation of the body frame at time t $R(t)$
  - Time rate of change $\dot{R}(t)$
  - Angular velocity $\omega_s \in \mathbb{R}^3$

$$\dot{r}_i = \omega_s \times r_i, \qquad i = 1, 2, 3.$$

Column

$$\dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3] = \omega_s \times R.$$

Skew-symmetric Matrix

$$x = [x_1 \ x_2 \ x_3]^{\mathrm{T}} \in \mathbb{R}^3 \qquad [x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$$[x] = -[x]^{\mathrm{T}}$$

$$\omega_s \times R = [\omega_s]R \qquad [\omega_s]R = \dot{R} \qquad [\omega_s] = \dot{R}R^{-1}$$

Proposition $$R[\omega]R^{\mathrm{T}} = [R\omega] \quad \omega \in \mathbb{R}^3 \quad R \in SO(3)$$

https://en.wikipedia.org/wiki/Skew-symmetric_matrix

See Lynch & Park for proof

Angular Velocities

- To express these equations in coordinates, we have to choose a reference frame for w
  - Two natural choices: fixed frame {s} or body frame {b} $[\omega_s] = \dot{R}R^{-1}$
- Consider body frame {b} ω_b

$$\omega_s = R_{sb}\omega_b \qquad \omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^{\mathrm{T}}\omega_s$$

$$\begin{aligned} [\omega_b] &= [R^{\mathrm{T}}\omega_s] &&= R^{\mathrm{T}}(\dot{R}R^{\mathrm{T}})R \\ &= R^{\mathrm{T}}[\omega_s]R &&= R^{\mathrm{T}}\dot{R} = R^{-1}\dot{R} \end{aligned}$$

Angular Velocities

- Orientation of the body frame at time t in the fixed frame $R(t)$

- Angular velocity w

$$\dot{R}R^{-1} = [\omega_s]$$
$$R^{-1}\dot{R} = [\omega_b]$$

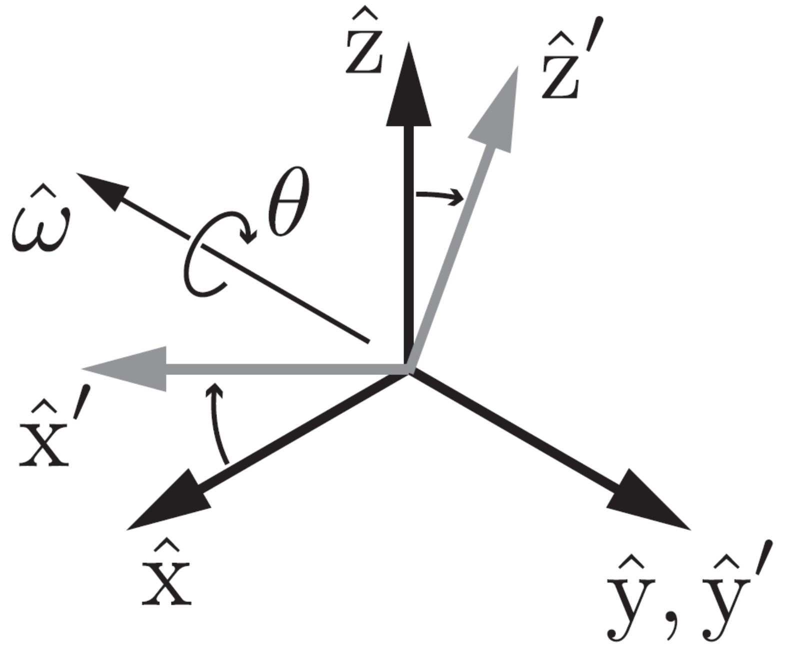
- Change of reference frame of angular velocity

$$\omega_c = R_{cd}\omega_d$$

Exponential Coordinate Representation of Rotation

- Exponential coordinates
  - A rotation axis (unit length) $\hat{\omega}$
  - An angle of rotation about the axis θ

$$\hat{\omega}\theta \in \mathbb{R}^3$$



- Interpretation $R = \mathrm{Rot}(\hat{\omega}, \theta)$
  - Axis-angle rotation of the fixed frame {s}
  - Apply angular velocity $\hat{\omega}\theta$ for one unit of time
  - Apply angular velocity $\hat{\omega}$ for θ units of time

Linear Differential Equations

- A differential equation is an equation that relates one or more functions and their derivatives

$$\frac{d\mathbf{x}_p(t)}{dt} = \dot{\mathbf{x}}_p(t) = \mathbf{v}(\mathbf{x}_p, t)$$

- A scalar linear differential equation $\dot{x}(t) = ax(t)$ $x(t) \in \mathbb{R},\ a \in \mathbb{R}$

Initial condition $x(0) = x_0$ Solution $x(t) = e^{at}x_0$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

Linear Differential Equations

- Vector linear differential equation

$$\dot{x}(t) = Ax(t) \qquad x(t) \in \mathbb{R}^n,\ A \in \mathbb{R}^{n \times n}$$

Initial condition $x(0) = x_0$ Solution $x(t) = e^{At}x_0$

matrix exponential $$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

If A is constant and finite, this series converges to a finite limit

Linear Differential Equations

- Vector linear differential equation

$$\dot{x}(t) = Ax(t) \qquad x(t) = e^{At}x_0$$

$$\begin{aligned}
\dot{x}(t) &= \left(\frac{d}{dt}e^{At}\right)x_0 \\
&= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right)x_0 \\
&= \left(A + A^2t + \frac{A^3t^2}{2!} + \cdots\right)x_0 \\
&= Ae^{At}x_0 \\
&= Ax(t),
\end{aligned}$$

Proof

Properties of the Matrix Exponential

1. $$d(e^{At})/dt = Ae^{At} = e^{At}A$$

2. $\textit{If } A = PDP^{-1} \textit{ for some } D \in \mathbb{R}^{n\times n} \textit{ and invertible } P \in \mathbb{R}^{n\times n} \textit{ then } e^{At} = Pe^{Dt}P^{-1}.$

3. $$\textit{If } AB = BA \textit{ then } e^Ae^B = e^{A+B}$$

4. $$(e^A)^{-1} = e^{-A}$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

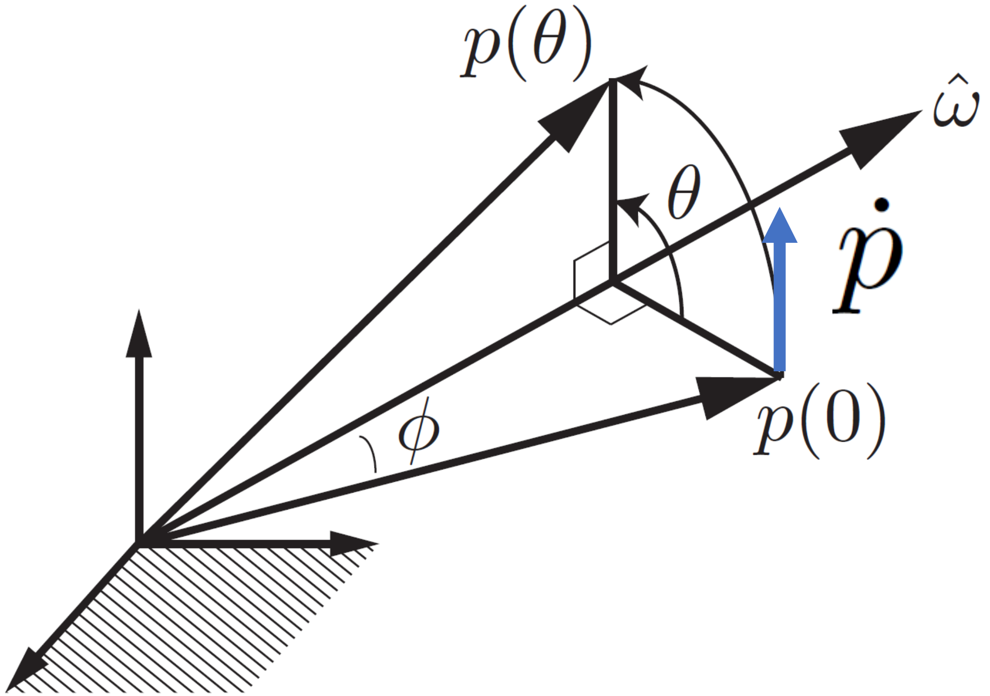
Exponential Coordinates

- p(0) is rotated to $p(\theta)$
  - At a constant rate of 1 rad/s
- p(t): path traced by the tip of vector

Velocity $$\dot{p} = \hat{\omega} \times p$$

Skew-symmetric Matrix $$\dot{p} = [\hat{\omega}]p$$

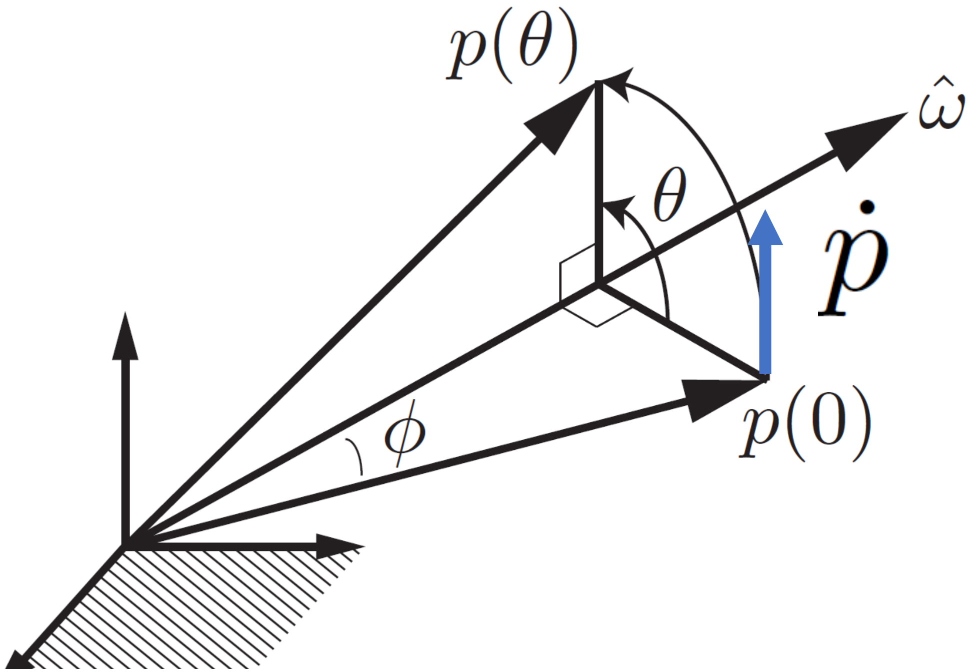
Vector linear differential equations $$p(t) = e^{[\hat{\omega}]t}p(0)$$



Exponential Coordinates

- $$p(t) = e^{[\hat{\omega}]t} p(0)$$

$$p(\theta) = e^{[\hat{\omega}]\theta} p(0)$$



$$[\hat{\omega}]^3 = -[\hat{\omega}]$$

$$\begin{aligned} e^{[\hat{\omega}]\theta} &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots \\ &= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat{\omega}]^2 \end{aligned}$$

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \end{aligned}$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

Rodrigues' formula: exponential coordinates to rotation matrix

Rodrigues' formula

$$\text{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)$$

$$\text{Rot}(\hat{\omega}, \theta) =$$

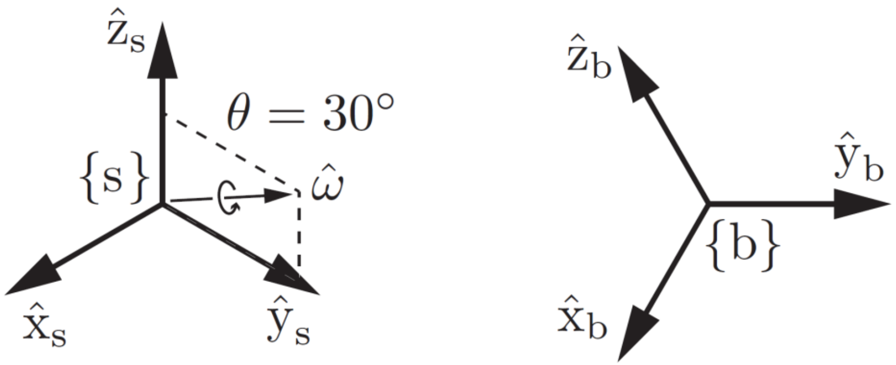
$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta \quad c_\theta = \cos\theta \quad \hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$$

Exponential Coordinates

- An example

$$\hat{\omega}_1 = (0, 0.866, 0.5) \qquad \theta_1 = 30^\circ$$



$$\begin{aligned} R &= e^{[\hat{\omega}_1]\theta_1} \\ &= I + \sin\theta_1 [\hat{\omega}_1] + (1 - \cos\theta_1)[\hat{\omega}_1]^2 \\ &= I + 0.5 \begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix} + 0.134 \begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0.866 & -0.250 & 0.433 \\ 0.250 & 0.967 & 0.058 \\ -0.433 & 0.058 & 0.899 \end{bmatrix}. \end{aligned}$$

Exponential Coordinates $\hat{\omega}_1 \theta_1 = (0, 0.453, 0.262)$

Summary

- Angular velocity

- Exponential coordinates

Further Reading

- Chapter 3 and Appendix B in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017

- Quaternion and Rotations, Yan-Bin Jia, https://graphics.stanford.edu/courses/cs348a-17-winter/Papers/quaternion.pdf

- Introduction to Robotics, Prof. Wei Zhang, OSU, Lecture 3, Rotational Motion, http://www2.ece.ohio-state.edu/~zhang/RoboticsClass/index.html