Wheeled Mobile Robots

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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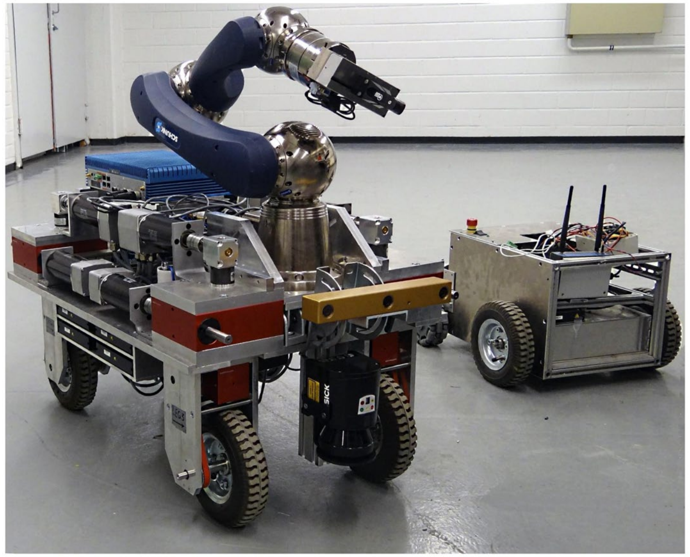
Wheeled Mobile Robots

https://stanleyinnovation.com/products-services/robotics/robotic-mobility-platforms/



https://www.mdpi.com/1424-8220/21/22/7642

https://ozrobotics.com/shop/3wd-100mm-omni-wheel-mobile-arduino-robot-kit-10013/

Wheeled Mobile Robots

- Kinematic model of a wheeled mobile robot
  - A chassis-fixed frame {b} relative to a fixed space frame {s} $T_{sb} \in SE(2)$
  - Represent T_{sb} by three coordinates $q = (\phi, x, y)$
  - Velocity of the chassis $\dot{q} = (\dot{\phi}, \dot{x}, \dot{y})$
  - Chassis' planar twist $\mathcal{V}_b = (\omega_{bz}, v_{bx}, v_{by})$

$$
\begin{aligned}
T^{-1}\dot{T} &= \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} R^{\mathrm{T}}\dot{R} & R^{\mathrm{T}}\dot{p} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

$$R^{\mathrm{T}}\dot{p} = v_b$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_{bz} \\ v_{bx} \\ v_{by} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \omega_{bz} \\ v_{bx} \\ v_{by} \end{bmatrix}$$

Types of Wheeled Mobile Robots

- Omindirectional
  - No equality constraints on the chassis velocity $\dot{q} = (\dot{\phi}, \dot{x}, \dot{y})$
  - Omniwheels

- Nonholonomic
  - Pfaffian constraints $A(q)\dot{q} = 0$
  - Car-like robots
  - Conventional wheels

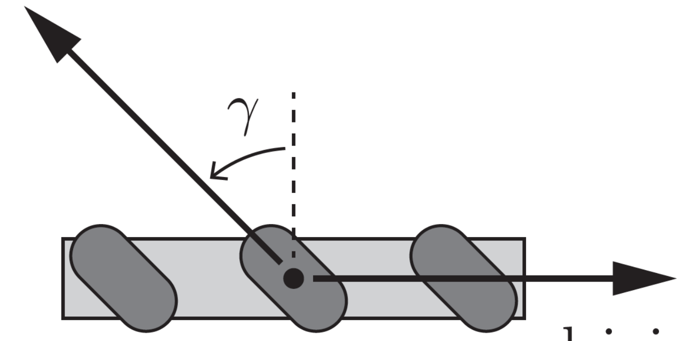
Omnidirectional Wheeled Mobile Robots

- Linear velocity of the center of the wheel in a frame at the wheel

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = v_{\text{drive}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_{\text{slide}} \begin{bmatrix} -\sin\gamma \\ \cos\gamma \end{bmatrix}$$

$$v_{\text{drive}} = v_x + v_y \tan\gamma,$$

$$v_{\text{slide}} = v_y / \cos\gamma.$$



Driving angular speed $$u = \frac{v_{\text{drive}}}{r} = \frac{1}{r}(v_x + v_y \tan\gamma)$$

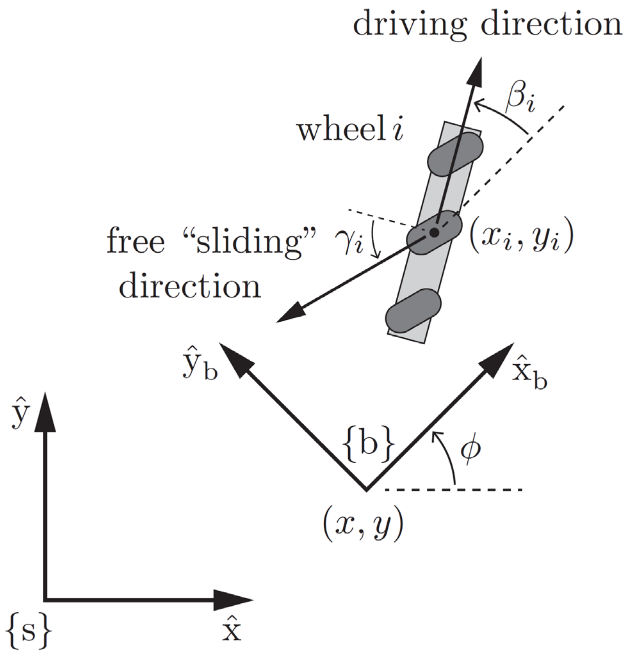
$$v = ru$$

Omnidirectional Wheeled Mobile Robots

$$u_i = h_i(\phi)\dot{q} =$$

wheel frame — Cross product

$$\begin{bmatrix} \frac{1}{r_i} & \frac{\tan\gamma_i}{r_i} \end{bmatrix} \begin{bmatrix} \cos\beta_i & \sin\beta_i \\ -\sin\beta_i & \cos\beta_i \end{bmatrix} \begin{bmatrix} -y_i & 1 & 0 \\ x_i & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$



$$\mathcal{V}_b = \begin{bmatrix} \omega_{bz} \\ v_{bx} \\ v_{by} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

$$h_i(\phi) = \frac{1}{r_i \cos\gamma_i} \begin{bmatrix} x_i \sin(\beta_i + \gamma_i) - y_i \cos(\beta_i + \gamma_i) \\ \cos(\beta_i + \gamma_i + \phi) \\ \sin(\beta_i + \gamma_i + \phi) \end{bmatrix}^{\mathrm{T}}$$

Omnidirectional Wheeled Mobile Robots

- For m omnidirectional wheels

Driving angular speed

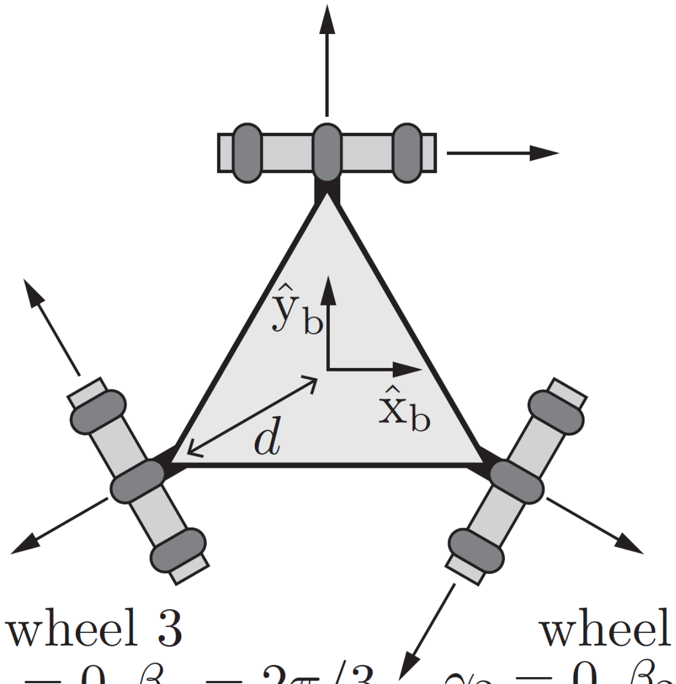
$$u = H(\phi)\dot{q} = \begin{bmatrix} h_1(\phi) \\ h_2(\phi) \\ \vdots \\ h_m(\phi) \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

$$H(\phi) \in \mathbb{R}^{m\times 3}$$

$$u = H(0)\mathcal{V}_b = \begin{bmatrix} h_1(0) \\ h_2(0) \\ \vdots \\ h_m(0) \end{bmatrix} \begin{bmatrix} \omega_{bz} \\ v_{bx} \\ v_{by} \end{bmatrix}$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_{bz} \\ v_{bx} \\ v_{by} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

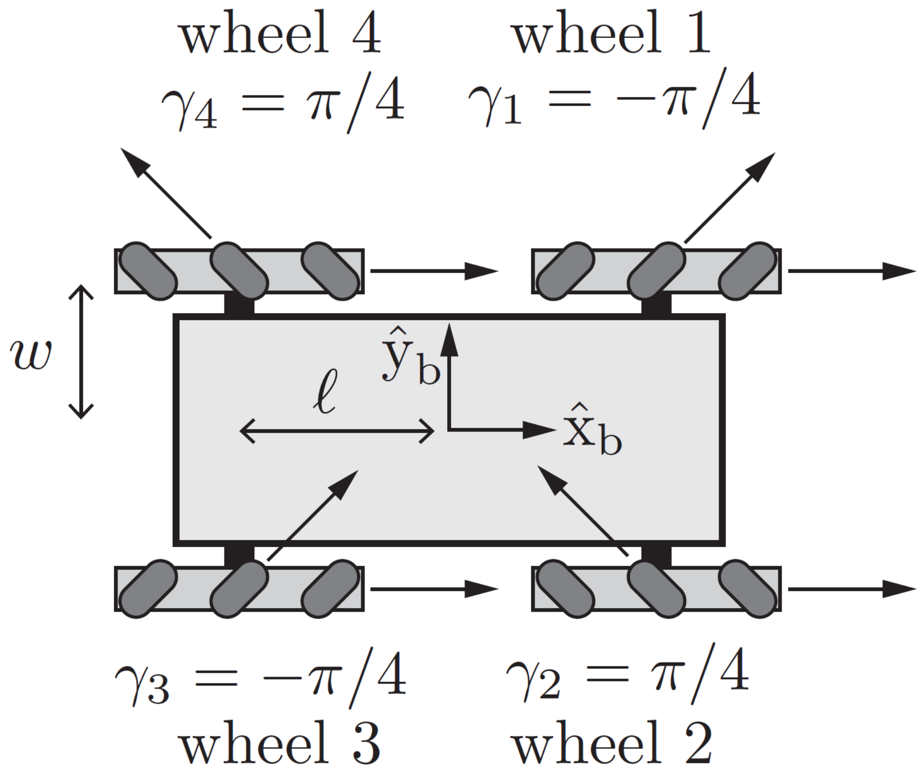
Omnidirectional Wheeled Mobile Robots



$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = H(0)\mathcal{V}_b$$

$$= \frac{1}{r}\begin{bmatrix} -d & 1 & 0 \\ -d & -1/2 & -\sin(\pi/3) \\ -d & -1/2 & \sin(\pi/3) \end{bmatrix} \begin{bmatrix} \omega_{bz} \\ v_{bx} \\ v_{by} \end{bmatrix}$$

Omnidirectional Wheeled Mobile Robots



$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = H(0)\mathcal{V}_b$$

$$= \frac{1}{r} \begin{bmatrix} -\ell - w & 1 & -1 \\ \ell + w & 1 & 1 \\ \ell + w & 1 & -1 \\ -\ell - w & 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_{bz} \\ v_{bx} \\ v_{by} \end{bmatrix}$$

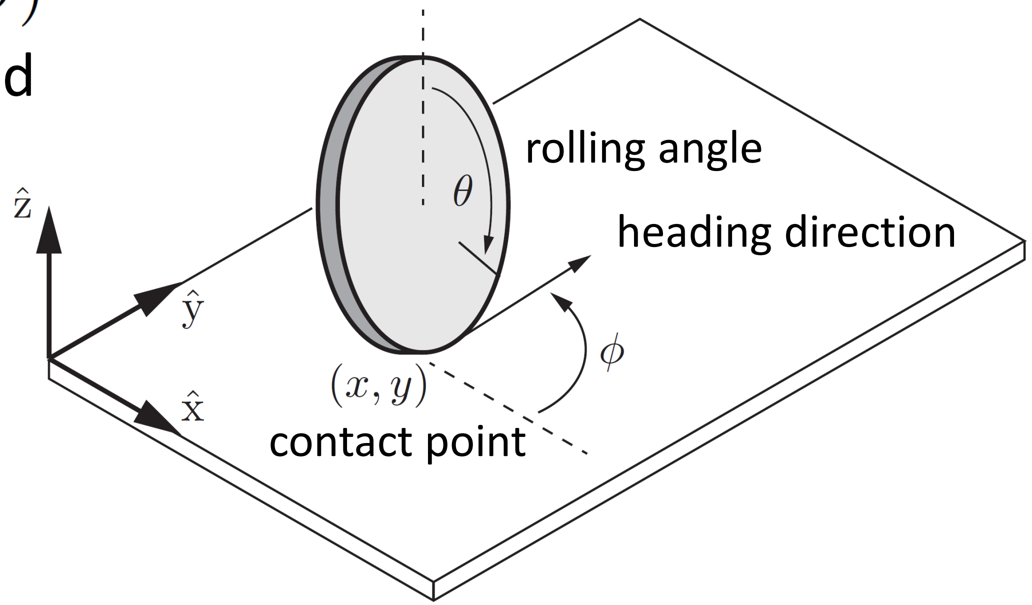
Nonholonomic Wheeled Mobile Robots

- The unicycle
  - Configuration of the wheel $q = (\phi, x, y, \theta)$
  - Control input: forward-backward driving speed u1 and heading direction turning speed u2
  - Kinematic equations of motion

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ r\cos\phi & 0 \\ r\sin\phi & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= G(q)u = g_1(q)u_1 + g_2(q)u_2$$

g1, g2 are called tangent vector fields
(control vector fields, velocity vector fields)



$$-u_{1,\max} \leq u_1 \leq u_{1,\max}$$

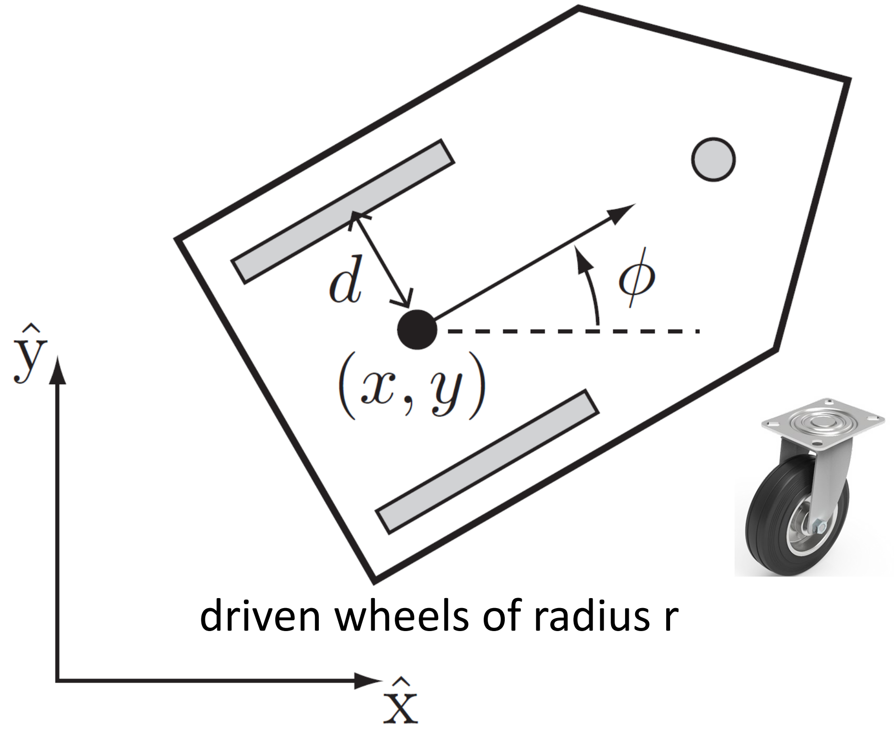
$$-u_{2,\max} \leq u_2 \leq u_{2,\max}$$

Nonholonomic Wheeled Mobile Robots

- The Differential-Drive Robot (diff-drive)

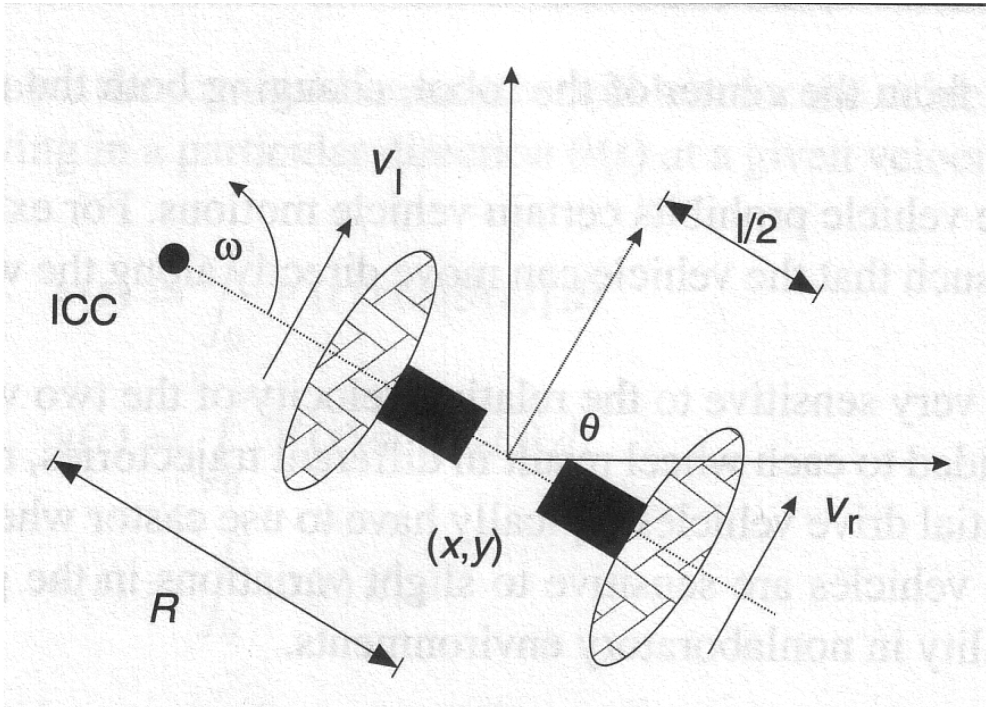
  - Two independently driven wheels of radius r that rotate about the same axis

  - One or more caster wheels, ball casters or low-friction slides that keep the car horizontal



Nonholonomic Wheeled Mobile Robots

- The Differential-Drive Robot



(a)

$$\omega\,(R + l/2) = V_r$$
$$\omega\,(R - l/2) = V_l$$

$$R = \frac{l}{2}\frac{V_l + V_r}{V_r - V_l};\quad \omega = \frac{V_r - V_l}{l};$$

https://www.cs.columbia.edu/~allen/F17/NOTES/icckinematics.pdf

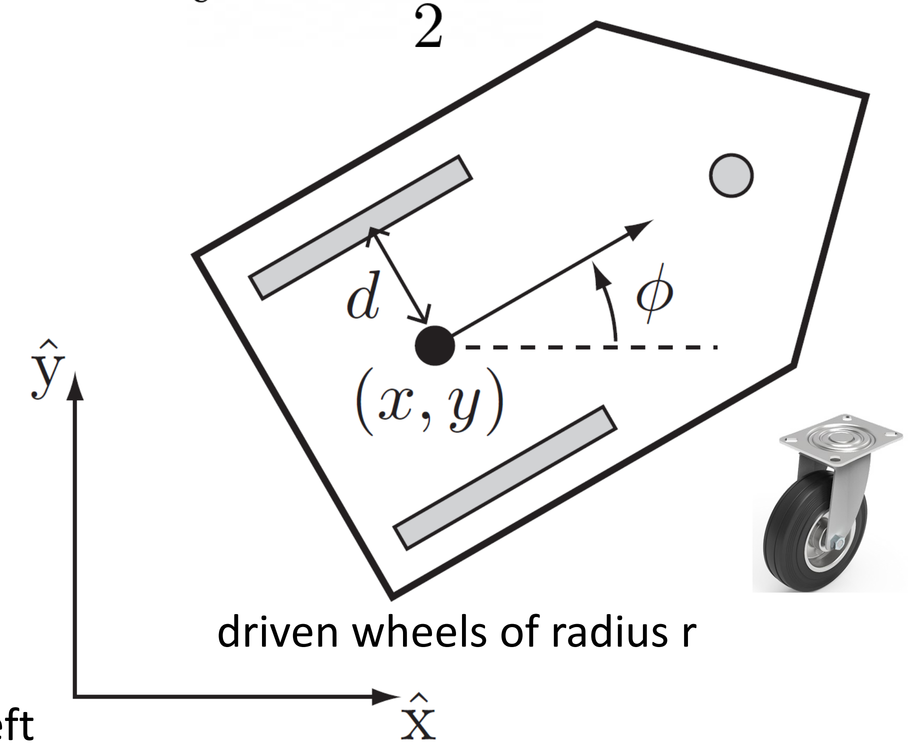
Nonholonomic Wheeled Mobile Robots

- Kinematic equations

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \\ \dot{\theta}_{\mathrm{L}} \\ \dot{\theta}_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} -r/2d & r/2d \\ \frac{r}{2}\cos\phi & \frac{r}{2}\cos\phi \\ \frac{r}{2}\sin\phi & \frac{r}{2}\sin\phi \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{\mathrm{L}} \\ u_{\mathrm{R}} \end{bmatrix}$$

angular speed of the left wheel and the right wheel

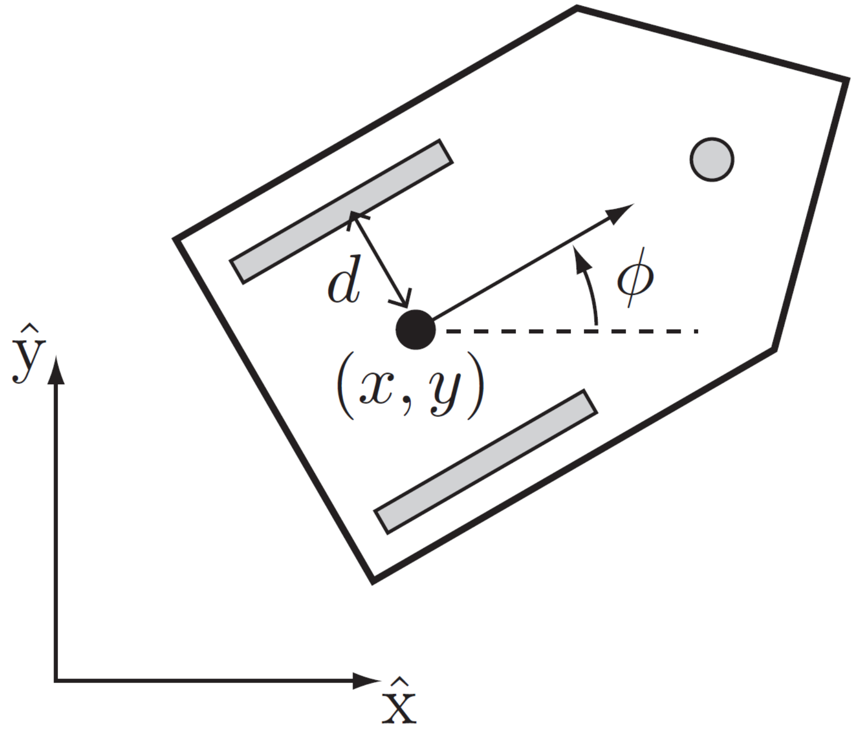
$$v = \frac{v_L + v_R}{2}$$



Nonholonomic Wheeled Mobile Robots

- Diff-drive

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r/2d & r/2d \\ \frac{r}{2}\cos\phi & \frac{r}{2}\cos\phi \\ \frac{r}{2}\sin\phi & \frac{r}{2}\sin\phi \end{bmatrix} \begin{bmatrix} u_L \\ u_R \end{bmatrix}$$

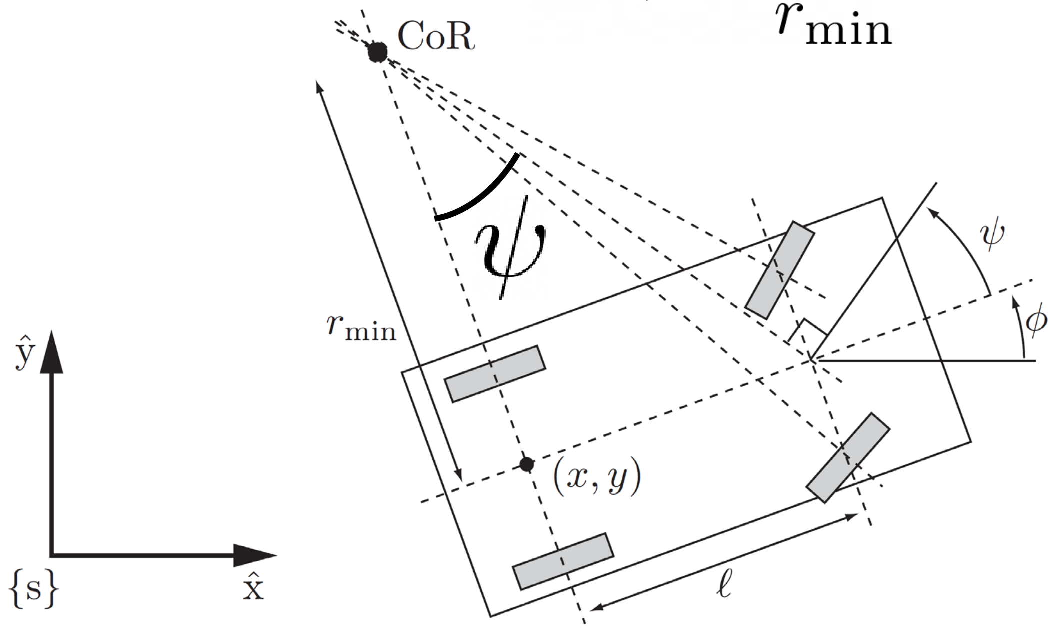


Nonholonomic Wheeled Mobile Robots

- The car-like robot
  - Configuration $q = (\phi, x, y, \psi)$
  - Heading direction ϕ
  - Steering angle ψ
  - Control inputs: forward speed v and angular speed w of the steering angle
- Kinematics

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} (\tan\psi)/\ell & 0 \\ \cos\phi & 0 \\ \sin\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

$$tan\psi = \frac{l}{r_{\min}}$$



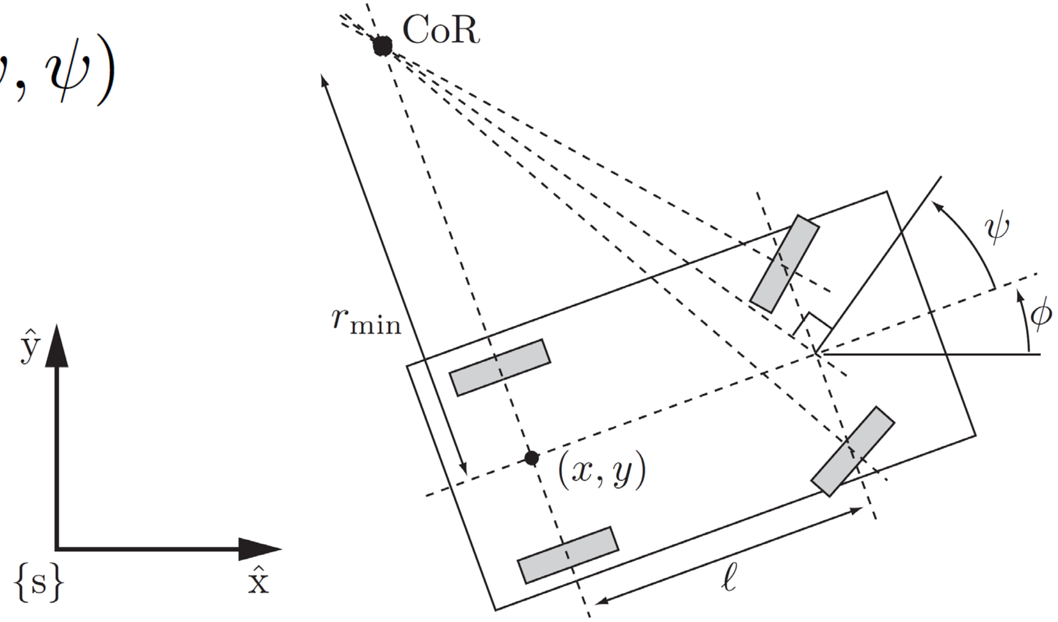
Nonholonomic Wheeled Mobile Robots

- The car-like robot
  - Simplify the control to steering angle ψ
  - Control inputs (v, ω) can be converted to (v, ψ)

Forward speed (v)   Rate of rotation (ω)

$$\psi = \tan^{-1}\left(\frac{\ell\omega}{v}\right)$$

$$v = r_{\min}\omega$$



Simplified car kinematics

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = G(q)u = \begin{bmatrix} 0 & 1 \\ \cos\phi & 0 \\ \sin\phi & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Nonholonomic Wheeled Mobile Robots

- Car kinematics

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = G(q)u = \begin{bmatrix} 0 & 1 \\ \cos\phi & 0 \\ \sin\phi & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

- Nonholonomic constraint

$$\dot{x} = v\cos\phi,$$

$$\dot{y} = v\sin\phi,$$

$$A(q)\dot{q} = [0 \;\; \sin\phi \;\; -\cos\phi]\dot{q} = \dot{x}\sin\phi - \dot{y}\cos\phi = 0.$$

Nonholonomic Wheeled Mobile Robots

- Canonical simplified model
  - Unicycle

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ r\cos\phi & 0 \\ r\sin\phi & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad u_1 = \frac{v}{r}, \qquad u_2 = \omega$$

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = G(q)u = \begin{bmatrix} 0 & 1 \\ \cos\phi & 0 \\ \sin\phi & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Nonholonomic Wheeled Mobile Robots

- Canonical simplified model
  - Diff-drive

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \\ \dot{\theta}_{\mathrm{L}} \\ \dot{\theta}_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} -r/2d & r/2d \\ \frac{r}{2}\cos\phi & \frac{r}{2}\cos\phi \\ \frac{r}{2}\sin\phi & \frac{r}{2}\sin\phi \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{\mathrm{L}} \\ u_{\mathrm{R}} \end{bmatrix}$$

$$u_{\mathrm{L}} = \frac{v - \omega d}{r}$$

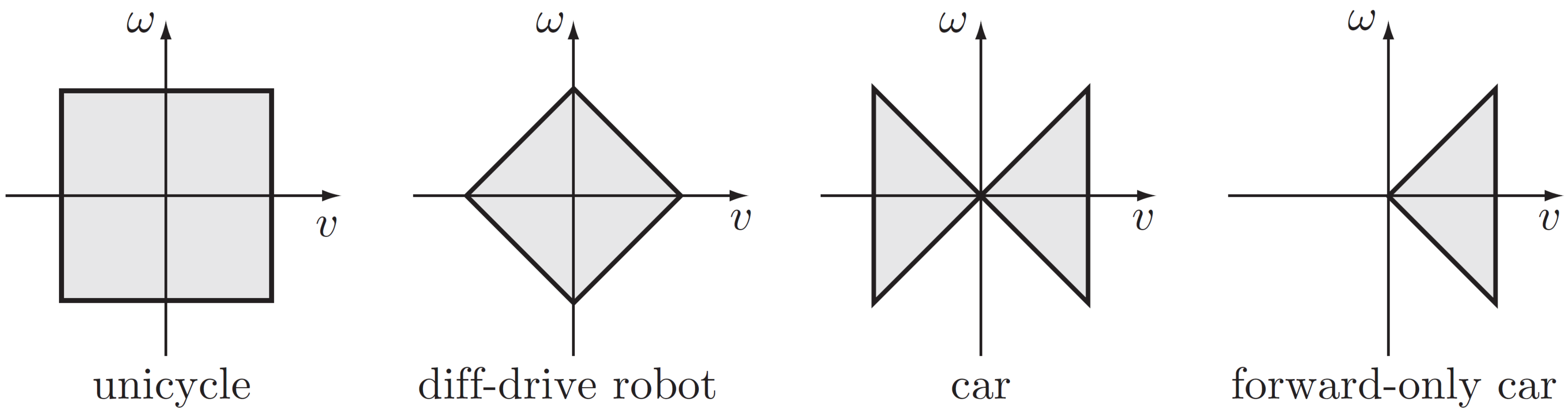
$$u_{\mathrm{R}} = \frac{v + \omega d}{r}$$

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = G(q)u = \begin{bmatrix} 0 & 1 \\ \cos\phi & 0 \\ \sin\phi & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Nonholonomic Wheeled Mobile Robots

- Canonical simplified model

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = G(q)u = \begin{bmatrix} 0 & 1 \\ \cos\phi & 0 \\ \sin\phi & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$



Odometry

- The process of estimating the chassis configuration q from the wheel motions
  - Wheel rotation sensing is available on all mobile robots
  - Estimation errors tend to accumulate over time
  - Supplement odometry with other position sensors such as GPS, laser, etc.

Odometry

- Estimate the new chassis configuration q_{k+1} given the previous configuration q_k and the change in wheel angles $\Delta\theta$

- Let $\Delta\theta_i$ be the change in wheel i's driving angle

$$\dot{\theta}_i = \Delta\theta_i / \Delta t \qquad \text{set } \Delta t = 1 \qquad \dot{\theta}_i = \Delta\theta_i$$

- For omnidirectional mobile robots

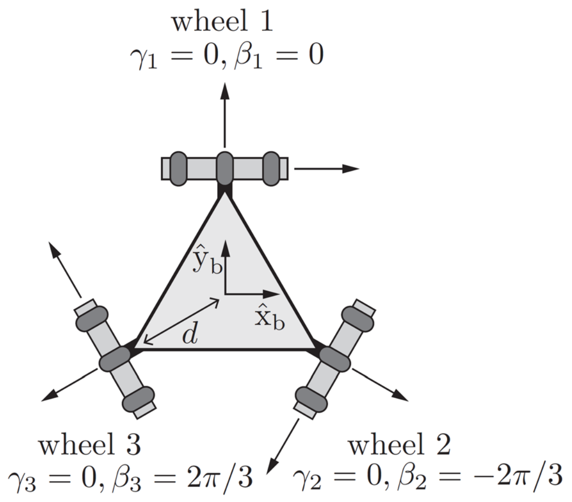
$$\Delta\theta = H(0)\mathcal{V}_b$$

$$\mathcal{V}_b = H^{\dagger}(0)\Delta\theta = F\Delta\theta$$

Odometry

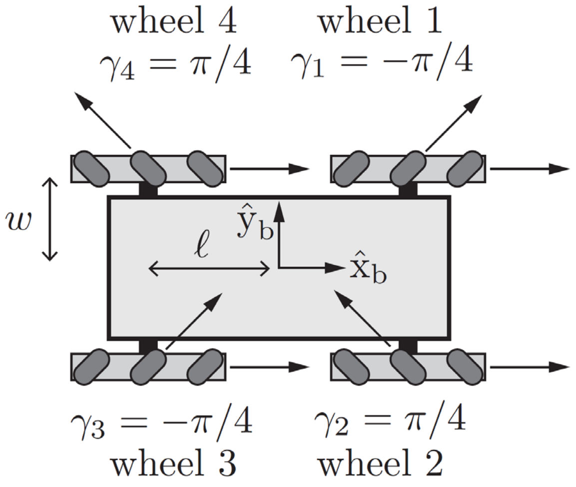
- For three-omniwheel robot

$$\mathcal{V}_b = F\Delta\theta = r\begin{bmatrix} -1/(3d) & -1/(3d) & -1/(3d) \\ 2/3 & -1/3 & -1/3 \\ 0 & -1/(2\sin(\pi/3)) & 1/(2\sin(\pi/3)) \end{bmatrix}\Delta\theta$$



- For four-mecanum-wheel robot

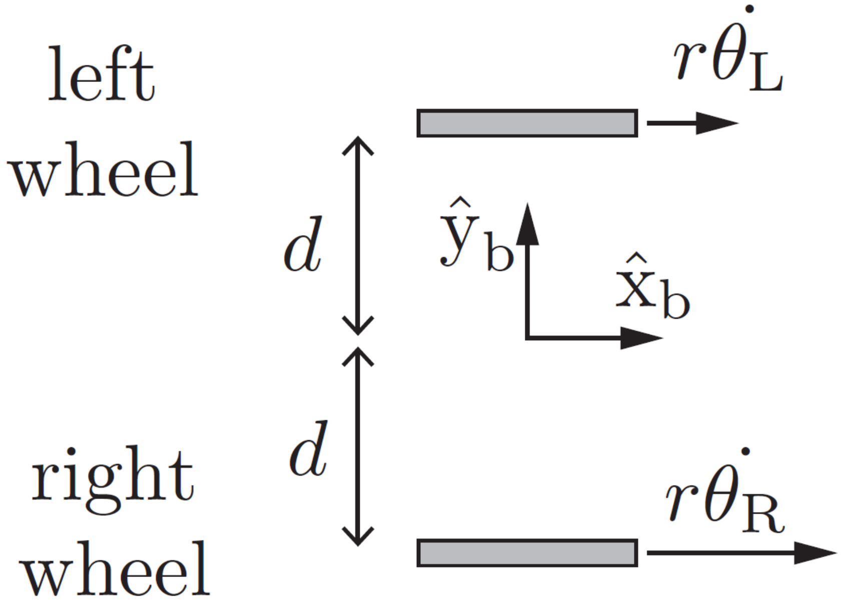
$$\mathcal{V}_b = F\Delta\theta = \frac{r}{4}\begin{bmatrix} -1/(\ell+w) & 1/(\ell+w) & 1/(\ell+w) & -1/(\ell+w) \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}\Delta\theta$$



Odometry

- Diff-drive robot

$$\mathcal{V}_b = F\Delta\theta = r\begin{bmatrix} -1/(2d) & 1/(2d) \\ 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Delta\theta_{\mathrm{L}} \\ \Delta\theta_{\mathrm{R}} \end{bmatrix}$$



Odometry

$$\mathcal{V}_b = F\Delta\theta$$

$$\mathcal{V}_{b6} = (0, 0, \omega_{bz}, v_{bx}, v_{by}, 0)$$

integrated to generate the displacement

$$T_{bb'} = e^{[\mathcal{V}_{b6}]}$$

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$$

$$\text{if } \ \omega_{bz} = 0, \qquad \Delta q_b = \begin{bmatrix} \Delta\phi_b \\ \Delta x_b \\ \Delta y_b \end{bmatrix} = \begin{bmatrix} 0 \\ v_{bx} \\ v_{by} \end{bmatrix}$$

$$\text{if } \ \omega_{bz} \neq 0, \qquad \Delta q_b = \begin{bmatrix} \Delta\phi_b \\ \Delta x_b \\ \Delta y_b \end{bmatrix} = \begin{bmatrix} \omega_{bz} \\ (v_{bx}\sin\omega_{bz} + v_{by}(\cos\omega_{bz} - 1))/\omega_{bz} \\ (v_{by}\sin\omega_{bz} + v_{bx}(1 - \cos\omega_{bz}))/\omega_{bz} \end{bmatrix}$$

Odometry

- In fixed frame {s}

$$\Delta q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_k & -\sin\phi_k \\ 0 & \sin\phi_k & \cos\phi_k \end{bmatrix} \Delta q_b$$

- Updated odometry

$$q_{k+1} = q_k + \Delta q$$

Summary

- Omnidirectional Wheeled Mobile Robots
- Nonholonomic Wheeled Mobile Robots
- Odometry

Further Reading

- Chapter 13 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.

- G. Oriolo. Wheeled robots. In J. Baillieul and T. Samad, editors, Encyclopedia of Systems and Control. Springer-Verlag, 2015.