Dynamics of Open Chains: Newton-Euler Formulation

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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Robot Dynamics

- Study motion of robots with the forces and torques that case them

- Equations of motion
  - A set of second-order differential equations

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

Joint variables $\theta \in \mathbb{R}^n$

Joint forces and torques $\tau \in \mathbb{R}^n$

$M(\theta) \in \mathbb{R}^{n \times n}$ a symmetric positive-definite mass matrix

$h(\theta, \dot{\theta}) \in \mathbb{R}^n$ forces that lump together centripetal, Coriolis, gravity, and friction terms that depend on θ and $\dot{\theta}$

Forward and Inverse Dynamics

- Forward dynamics
  - Given robot state $(\theta, \dot{\theta})$ and the joint forces and torques
  - Determine the robot's acceleration

$$\ddot{\theta} = M^{-1}(\theta)\left(\tau - h(\theta, \dot{\theta})\right)$$

- Inverse dynamics
  - Given robot state $(\theta, \dot{\theta})$ and a desired acceleration
  - Find the joint forces and torques

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

Robot Dynamics

- Lagrangian formulation
  - Kinetic energy and potential energy

- Newton-Euler formulation
  - F = ma

  - Last lecture: a single rigid body

  - This lecture: a N-link open chain

Inverse Dynamics

- N-link open chain
- A body-fixed reference frame {i} is attached to the center of mass of each link i
- Base frame {0}, end-effector frame {n+1} (fixed in {n})
- At home position (all joints are zeros)
  - Configuration of frame {j} in {i} $M_{i,j} \in SE(3)$
  - Configuration of {i} in base frame {0} $M_i = M_{0,i}$

$$M_{i-1,i} = M_{i-1}^{-1} M_i \qquad M_{i,i-1} = M_i^{-1} M_{i-1}$$

Inverse Dynamics

- Screw axis for joint i in link frame {i} $\mathcal{A}_i$, in space frame {0} $\mathcal{S}_i$

$$\mathcal{A}_i = \mathrm{Ad}_{M_i^{-1}}(\mathcal{S}_i)$$

- Screw axis is a normalize twist

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6 \qquad \mathcal{S}\dot{\theta} = \mathcal{V}$$

$$\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b \qquad [\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Inverse Dynamics

- Screw axis for joint i in link frame {i} $\mathcal{A}_i$, in space frame {0} $\mathcal{S}_i$

$$\mathcal{A}_i = \mathrm{Ad}_{M_i^{-1}}(\mathcal{S}_i)$$

- The configuration of {j} in {i} with joint variables $T_{i,j} \in SE(3)$

$$T_{i-1,i}(\theta_i) \qquad T_{i,i-1}(\theta_i) = T_{i-1,i}^{-1}(\theta_i)$$

$$T_{i-1,i}(\theta_i) = M_{i-1,i}e^{[\mathcal{A}_i]\theta_i} \qquad T_{i,i-1}(\theta_i) = e^{-[\mathcal{A}_i]\theta_i}M_{i,i-1}$$

- Twist of link frame {i} $\mathcal{V}_i = (\omega_i, v_i)$
- Wrench transmitted through joint i to link frame {i} $\mathcal{F}_i = (m_i, f_i)$

Inverse Dynamics

- Spatial inertia matrix of link i $\mathcal{G}_i \in \mathbb{R}^{6\times 6}$ $\mathcal{G}_i = \begin{bmatrix} \mathcal{I}_i & 0 \\ 0 & \mathfrak{m}_i I \end{bmatrix}$

- Recursively calculate the twist and acceleration, moving from the base to the tip

$$\mathcal{V}_i = \mathcal{A}_i\dot{\theta}_i + [\mathrm{Ad}_{T_{i,i-1}}]\mathcal{V}_{i-1}$$

$$\dot{\mathcal{V}}_i = \mathcal{A}_i\ddot{\theta}_i + [\mathrm{Ad}_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + \frac{d}{dt}\left([\mathrm{Ad}_{T_{i,i-1}}]\right)\mathcal{V}_{i-1}$$

Inverse Dynamics

$$T_{i,i-1} = \begin{bmatrix} R_{i,i-1} & p \\ 0 & 1 \end{bmatrix}$$

Screw axis $\mathcal{A}_i = \begin{bmatrix} \omega \\ v \end{bmatrix}$

Recall

$$R^{-1}\dot{R} = [\omega_b]$$

$$[a] = -[a]^{\mathrm{T}}$$

$$[a]b = -[b]a$$

$$[a][b] = ([b][a])^{\mathrm{T}}$$

$$\frac{d}{dt}\left([\mathrm{Ad}_{T_{i,i-1}}]\right)\mathcal{V}_{i-1} = \frac{d}{dt}\left(\begin{bmatrix} R_{i,i-1} & 0 \\ [p]R_{i,i-1} & R_{i,i-1} \end{bmatrix}\right)\mathcal{V}_{i-1}$$

$$= \begin{bmatrix} -[\omega\dot{\theta}_i]R_{i,i-1} & 0 \\ -[v\dot{\theta}_i]R_{i,i-1} - [\omega\dot{\theta}_i][p]R_{i,i-1} & -[\omega\dot{\theta}_i]R_{i,i-1} \end{bmatrix}\mathcal{V}_{i-1}$$

$$= \underbrace{\begin{bmatrix} -[\omega\dot{\theta}_i] & 0 \\ -[v\dot{\theta}_i] & -[\omega\dot{\theta}_i] \end{bmatrix}}_{-[\mathrm{ad}_{\mathcal{A}_i\dot{\theta}_i}]}\underbrace{\begin{bmatrix} R_{i,i-1} & 0 \\ [p]R_{i,i-1} & R_{i,i-1} \end{bmatrix}}_{[\mathrm{Ad}_{T_{i,i-1}}]}\mathcal{V}_{i-1}$$

$$= -[\mathrm{ad}_{\mathcal{A}_i\dot{\theta}_i}]\mathcal{V}_i = [\mathrm{ad}_{\mathcal{V}_i}]\mathcal{A}_i\dot{\theta}_i$$

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

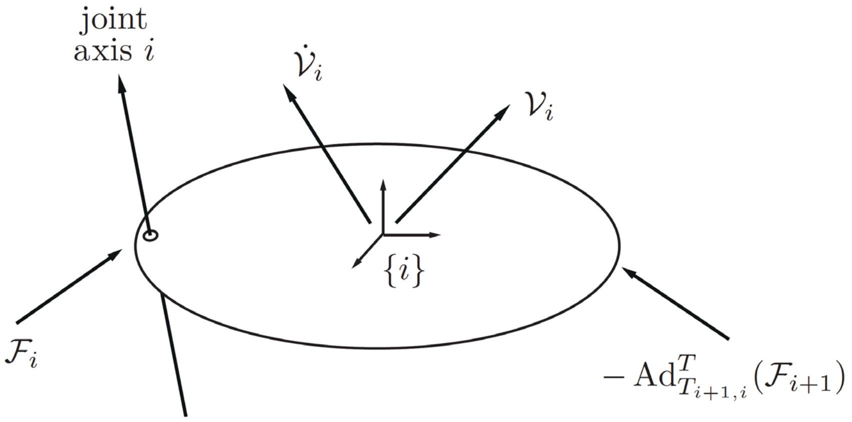
Inverse Dynamics

- Accelerations from base to tip

$$\dot{\mathcal{V}}_i = \mathcal{A}_i \ddot{\theta}_i + [\mathrm{Ad}_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + [\mathrm{ad}_{\mathcal{V}_i}]\mathcal{A}_i \dot{\theta}_i$$

- Recall rigid body dynamic equations

$$\begin{aligned} \mathcal{F}_b &= \mathcal{G}_b \dot{\mathcal{V}}_b - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_b}(\mathcal{P}_b) \\ &= \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b \end{aligned}$$



- Wrench on link i from joint i and joint i+1

$$\mathcal{G}_i \dot{\mathcal{V}}_i - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_i}(\mathcal{G}_i \mathcal{V}_i) = \mathcal{F}_i - \mathrm{Ad}^{\mathrm{T}}_{T_{i+1,i}}(\mathcal{F}_{i+1})$$

Inverse Dynamics

- Solve the wrench from tip to base $\mathcal{F}_i$

- Force or torque at the joint in the direction of the joint's screw axis

$$\tau_i = \mathcal{F}_i^{\mathrm{T}} \mathcal{A}_i$$

- Newton-Euler Inverse Dynamics Algorithm

Newton-Euler Inverse Dynamics Algorithm

Forward iterations Given $\theta, \dot{\theta}, \ddot{\theta}$, for $i = 1$ to n do

$$
\begin{aligned}
T_{i,i-1} &= e^{-[\mathcal{A}_i]\theta_i} M_{i,i-1}, \\
\mathcal{V}_i &= \mathrm{Ad}_{T_{i,i-1}}(\mathcal{V}_{i-1}) + \mathcal{A}_i \dot{\theta}_i, \\
\dot{\mathcal{V}}_i &= \mathrm{Ad}_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + \mathrm{ad}_{\mathcal{V}_i}(\mathcal{A}_i)\dot{\theta}_i + \mathcal{A}_i \ddot{\theta}_i.
\end{aligned}
$$

Backward iterations For $i = n$ to 1 do

$$
\begin{aligned}
\mathcal{F}_i &= \mathrm{Ad}^{\mathrm{T}}_{T_{i+1,i}}(\mathcal{F}_{i+1}) + \mathcal{G}_i \dot{\mathcal{V}}_i - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_i}(\mathcal{G}_i \mathcal{V}_i), \\
\tau_i &= \mathcal{F}_i^{\mathrm{T}} \mathcal{A}_i.
\end{aligned}
$$

Dynamics Equations in Closed Form

- Dynamic equations $\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta)$
- Definitions

$$\mathcal{V} = \begin{bmatrix} \mathcal{V}_1 \\ \vdots \\ \mathcal{V}_n \end{bmatrix} \in \mathbb{R}^{6n} \qquad \mathcal{F} = \begin{bmatrix} \mathcal{F}_1 \\ \vdots \\ \mathcal{F}_n \end{bmatrix} \in \mathbb{R}^{6n} \qquad \mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mathcal{A}_n \end{bmatrix} \in \mathbb{R}^{6n \times n}$$

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{G}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mathcal{G}_n \end{bmatrix} \in \mathbb{R}^{6n \times 6n} \qquad [\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\mathrm{ad}_{\mathcal{V}_1}] & 0 & \cdots & 0 \\ 0 & [\mathrm{ad}_{\mathcal{V}_2}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & [\mathrm{ad}_{\mathcal{V}_n}] \end{bmatrix} \in \mathbb{R}^{6n \times 6n}$$

$$[\mathrm{ad}_{\mathcal{A}\dot{\theta}}] = \begin{bmatrix} [\mathrm{ad}_{\mathcal{A}_1\dot{\theta}_1}] & 0 & \cdots & 0 \\ 0 & [\mathrm{ad}_{\mathcal{A}_2\dot{\theta}_2}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & [\mathrm{ad}_{\mathcal{A}_n\dot{\theta}_n}] \end{bmatrix} \in \mathbb{R}^{6n \times 6n} \qquad \mathcal{W}(\theta) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ [\mathrm{Ad}_{T_{21}}] & 0 & \cdots & 0 & 0 \\ 0 & [\mathrm{Ad}_{T_{32}}] & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & [\mathrm{Ad}_{T_{n,n-1}}] & 0 \end{bmatrix} \in \mathbb{R}^{6n \times 6n}$$

Dynamics Equations in Closed Form

$$\mathcal{V}_{\text{base}} = \begin{bmatrix} \mathrm{Ad}_{T_{10}}(\mathcal{V}_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n} \qquad \dot{\mathcal{V}}_{\text{base}} = \begin{bmatrix} \mathrm{Ad}_{T_{10}}(\dot{\mathcal{V}}_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n} \qquad \mathcal{F}_{\text{tip}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ad}^{\mathrm{T}}_{T_{n+1,n}}(\mathcal{F}_{n+1}) \end{bmatrix} \in \mathbb{R}^{6n}$$

Recursive inverse dynamics algorithm

$$\begin{aligned}
\mathcal{V} &= \mathcal{W}(\theta)\mathcal{V} + \mathcal{A}\dot{\theta} + \mathcal{V}_{\text{base}}, \\
\dot{\mathcal{V}} &= \mathcal{W}(\theta)\dot{\mathcal{V}} + \mathcal{A}\ddot{\theta} - [\mathrm{ad}_{\mathcal{A}\dot{\theta}}](\mathcal{W}(\theta)\mathcal{V} + \mathcal{V}_{\text{base}}) + \dot{\mathcal{V}}_{\text{base}}, \\
\mathcal{F} &= \mathcal{W}^{\mathrm{T}}(\theta)\mathcal{F} + \mathcal{G}\dot{\mathcal{V}} - [\mathrm{ad}_{\mathcal{V}}]^{\mathrm{T}}\mathcal{G}\mathcal{V} + \mathcal{F}_{\text{tip}}, \\
\tau &= \mathcal{A}^{\mathrm{T}}\mathcal{F}.
\end{aligned}$$

Dynamics Equations in Closed Form

- Define $\mathcal{L}(\theta) = (I - \mathcal{W}(\theta))^{-1}$

$$
\begin{aligned}
\mathcal{V} &= \mathcal{L}(\theta)\left(\mathcal{A}\dot{\theta} + \mathcal{V}_{\text{base}}\right), \\
\dot{\mathcal{V}} &= \mathcal{L}(\theta)\left(\mathcal{A}\ddot{\theta} + [\text{ad}_{\mathcal{A}\dot{\theta}}]\mathcal{W}(\theta)\mathcal{V} + [\text{ad}_{\mathcal{A}\dot{\theta}}]\mathcal{V}_{\text{base}} + \dot{\mathcal{V}}_{\text{base}}\right) \\
\mathcal{F} &= \mathcal{L}^T(\theta)\left(\mathcal{G}\dot{\mathcal{V}} - [\text{ad}_{\mathcal{V}}]^{\text{T}}\mathcal{G}\mathcal{V} + \mathcal{F}_{\text{tip}}\right), \\
\tau &= \mathcal{A}^T\mathcal{F}.
\end{aligned}
$$

Dynamics Equations in Closed Form

- If the robot applies an external wrench at the end-effector $\mathcal{F}_{\mathrm{tip}}$

End-effector torque $\tau = J^{\mathrm{T}}(\theta) f_{\mathrm{tip}}$

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) + J^{\mathrm{T}}(\theta)\mathcal{F}_{\mathrm{tip}}$$

$$\begin{aligned}
M(\theta) &= \mathcal{A}^T \mathcal{L}^{\mathrm{T}}(\theta)\mathcal{G}\mathcal{L}(\theta)\mathcal{A}, \\
c(\theta, \dot{\theta}) &= -\mathcal{A}^T \mathcal{L}^T(\theta)\left(\mathcal{G}\mathcal{L}(\theta)\left[\mathrm{ad}_{\mathcal{A}\dot{\theta}}\right]\mathcal{W}(\theta) + [\mathrm{ad}_{\mathcal{V}}]^{\mathrm{T}}\mathcal{G}\right)\mathcal{L}(\theta)\mathcal{A}\dot{\theta}, \\
g(\theta) &= \mathcal{A}^T \mathcal{L}^T(\theta)\mathcal{G}\mathcal{L}(\theta)\dot{\mathcal{V}}_{\mathrm{base}}.
\end{aligned}$$

Forward Dynamics of Open Chains

- Forward dynamics $M(\theta)\ddot{\theta} = \tau(t) - h(\theta, \dot{\theta}) - J^{\mathrm{T}}(\theta)\mathcal{F}_{\mathrm{tip}}$
  - Given $\theta, \ \dot{\theta}, \ \tau \ \mathcal{F}_{\mathrm{tip}}$ Solve $\ddot{\theta}$
- $h(\theta, \dot{\theta})$ can be computed by the inverse dynamics algorithm with $\ddot{\theta} = 0$ and $\mathcal{F}_{\mathrm{tip}} = 0$
- The inertia matrix $M(\theta) = \sum_{i=1}^{n} J_{ib}^{\mathrm{T}}(\theta)\mathcal{G}_i J_{ib}(\theta)$ $\qquad \mathcal{V}_i = J_{ib}(\theta)\dot{\theta}$
- We can solve

$$M\ddot{\theta} = b, \text{ for } \ddot{\theta}$$

Forward Dynamics of Open Chains

- Simulate the motion of a robot

$$\ddot{\theta} = ForwardDynamics(\theta, \dot{\theta}, \tau, \mathcal{F}_{\text{tip}})$$

First-order differential equations

$$q_1 = \theta, \; q_2 = \dot{\theta}$$

$$\begin{aligned} \dot{q}_1 &= q_2, \\ \dot{q}_2 &= ForwardDynamics(q_1, q_2, \tau, \mathcal{F}_{\text{tip}}) \end{aligned}$$

First-order Euler iteration

$$\begin{aligned} q_1(t + \delta t) &= q_1(t) + q_2(t)\delta t, \\ q_2(t + \delta t) &= q_2(t) + ForwardDynamics(q_1, q_2, \tau, \mathcal{F}_{\text{tip}})\delta t \end{aligned}$$

Initial values $q_1(0) = \theta(0)$ and $q_2(0) = \dot{\theta}(0)$

Summary

- Newton-Euler Inverse Dynamics Algorithm

- Forward Dynamics of Open Chains

Further Reading

- Chapter 8 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.