Dynamics of Open Chains: A Single Rigid Body

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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Robot Dynamics

- Study motion of robots with the forces and torques that case them

- Equations of motion
  - A set of second-order differential equations

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

Joint variables $\theta \in \mathbb{R}^n$

Joint forces and torques $\tau \in \mathbb{R}^n$

$M(\theta) \in \mathbb{R}^{n \times n}$ a symmetric positive-definite mass matrix

$h(\theta, \dot{\theta}) \in \mathbb{R}^n$ forces that lump together centripetal, Coriolis, gravity, and friction terms that depend on θ and $\dot{\theta}$

Forward and Inverse Dynamics

- Forward dynamics
  - Given robot state $(\theta, \dot{\theta})$ and the joint forces and torques
  - Determine the robot's acceleration

$$\ddot{\theta} = M^{-1}(\theta)\left(\tau - h(\theta, \dot{\theta})\right)$$

- Inverse dynamics
  - Given robot state $(\theta, \dot{\theta})$ and a desired acceleration
  - Find the joint forces and torques

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

Robot Dynamics

- Lagrangian formulation
  - Kinetic energy and potential energy

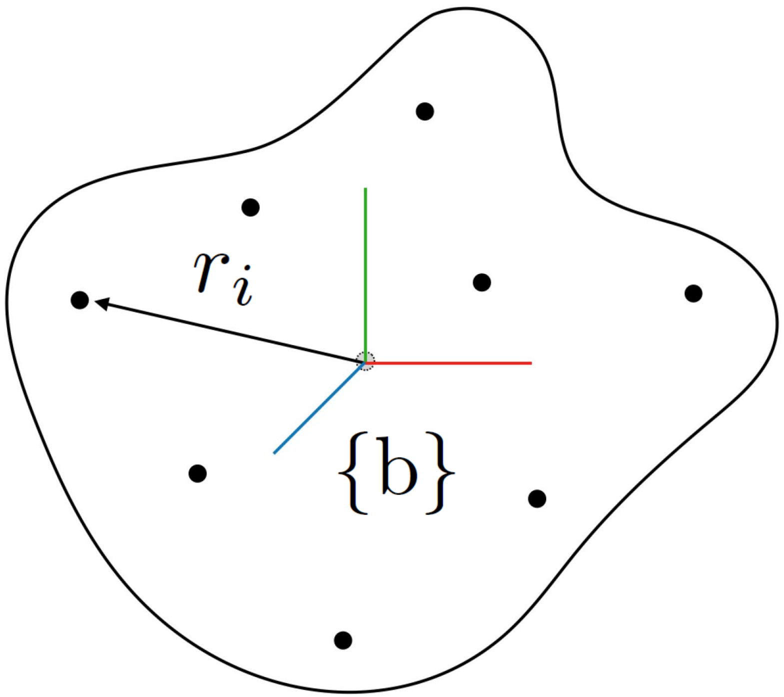
- Newton-Euler formulation
  - F = ma

Dynamics of a Single Rigid Body

- A rigid body with a set of point masses
- Total mass $\mathfrak{m} = \sum_i \mathfrak{m}_i$
- The origin of the body frame

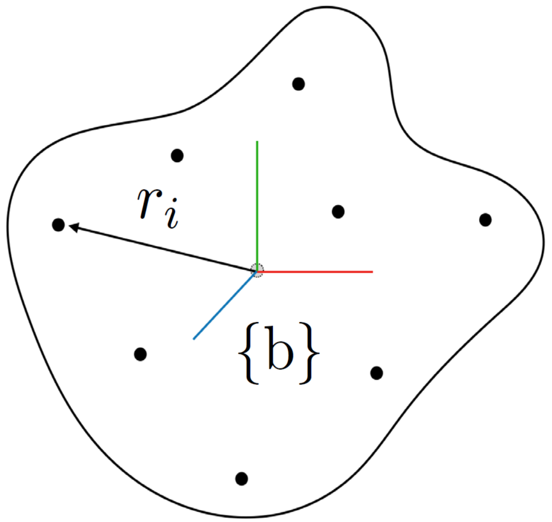
  Center of mass $\sum_i \mathfrak{m}_i r_i = 0$

- If some other point is chosen as origin, move the origin to $(1/\mathfrak{m}) \sum_i \mathfrak{m}_i r_i$



Dynamics of a Single Rigid Body

- Assume the body is moving with a body twist $\mathcal{V}_b = (\omega_b, v_b)$
- $p_i(t)$ be the time-varying position of $\mathfrak{m}_i$, initially at r_i



$$\dot{p}_i = v_b + \omega_b \times p_i$$

$$\ddot{p}_i = \dot{v}_b + \frac{d}{dt}\omega_b \times p_i + \omega_b \times \frac{d}{dt}p_i$$

$$= \dot{v}_b + \dot{\omega}_b \times p_i + \omega_b \times (v_b + \omega_b \times p_i)$$

$$\ddot{p}_i = \dot{v}_b + [\dot{\omega}_b]r_i + [\omega_b]v_b + [\omega_b]^2 r_i$$

Dynamics of a Single Rigid Body

- For a point mass $f_i = \mathfrak{m}_i \ddot{p}_i$

$$f_i = \mathfrak{m}_i(\dot{v}_b + [\dot{\omega}_b]r_i + [\omega_b]v_b + [\omega_b]^2 r_i)$$

- Moment of the point mass $m_i = [r_i]f_i$
- Total force and moment on the body

Wrench $\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \sum_i m_i \\ \sum_i f_i \end{bmatrix}$

Dynamics of a Single Rigid Body

- Linear dynamics

$$f_b = \sum_i \mathfrak{m}_i(\dot{v}_b + [\dot{\omega}_b]r_i + [\omega_b]v_b + [\omega_b]^2 r_i)$$

$$= \sum_i \mathfrak{m}_i(\dot{v}_b + [\omega_b]v_b) - \cancel{\sum_i \mathfrak{m}_i[r_i]}^{0}\dot{\omega}_b + \cancel{\sum_i \mathfrak{m}_i[r_i]}^{0}[\omega_b]\omega_b$$

$$= \sum_i \mathfrak{m}_i(\dot{v}_b + [\omega_b]v_b)$$

$$= \mathfrak{m}(\dot{v}_b + [\omega_b]v_b).$$

$$\textstyle\sum_i \mathfrak{m}_i[r_i] = 0$$

$$[a] = -[a]^{\mathrm{T}}$$

$$[a]b = -[b]a$$

$$[a][b] = ([b][a])^{\mathrm{T}}$$

Dynamics of a Single Rigid Body

- Rotational dynamics

$$
\begin{aligned}
m_b &= \sum_i \mathfrak{m}_i [r_i](\dot{v}_b + [\dot{\omega}_b] r_i + [\omega_b] v_b + [\omega_b]^2 r_i) \\
&= \sum_i \mathfrak{m}_i [r_i] \dot{v}_b \overset{0}{} + \sum_i \mathfrak{m}_i [r_i][\omega_b] v_b \overset{0}{} \\
&\quad + \sum_i \mathfrak{m}_i [r_i]([\dot{\omega}_b] r_i + [\omega_b]^2 r_i) \\
&= \sum_i \mathfrak{m}_i \left(-[r_i]^2 \dot{\omega}_b - [r_i]^{\mathrm{T}} [\omega_b]^{\mathrm{T}} [r_i] \omega_b\right) \\
&= \sum_i \mathfrak{m}_i \left(-[r_i]^2 \dot{\omega}_b - [\omega_b][r_i]^2 \omega_b\right) \\
&= \left(-\sum_i \mathfrak{m}_i [r_i]^2\right) \dot{\omega}_b + [\omega_b] \left(-\sum_i \mathfrak{m}_i [r_i]^2\right) \omega_b \\
&= \mathcal{I}_b \dot{\omega}_b + [\omega_b] \mathcal{I}_b \omega_b,
\end{aligned}
$$

Body's rotational inertia matrix

$$\mathcal{I}_b = -\sum_i \mathfrak{m}_i [r_i]^2 \in \mathbb{R}^{3\times 3}$$

Euler's equation for a rotating rigid body

Dynamics of a Single Rigid Body

- Rotational inertia matrix $\mathcal{I}_b = -\sum_i \mathfrak{m}_i [r_i]^2 \in \mathbb{R}^{3\times 3}$

$$\begin{aligned} \mathcal{I}_b &= \begin{bmatrix} \sum \mathfrak{m}_i(y_i^2+z_i^2) & -\sum \mathfrak{m}_i x_i y_i & -\sum \mathfrak{m}_i x_i z_i \\ -\sum \mathfrak{m}_i x_i y_i & \sum \mathfrak{m}_i(x_i^2+z_i^2) & -\sum \mathfrak{m}_i y_i z_i \\ -\sum \mathfrak{m}_i x_i z_i & -\sum \mathfrak{m}_i y_i z_i & \sum \mathfrak{m}_i(x_i^2+y_i^2) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \mathcal{I}_{xx} &= \int_{\mathcal{B}} (y^2+z^2)\rho(x,y,z)\,dV & \mathcal{I}_{xy} &= -\int_{\mathcal{B}} xy\rho(x,y,z)\,dV \\ \mathcal{I}_{yy} &= \int_{\mathcal{B}} (x^2+z^2)\rho(x,y,z)\,dV & \mathcal{I}_{xz} &= -\int_{\mathcal{B}} xz\rho(x,y,z)\,dV \\ \mathcal{I}_{zz} &= \int_{\mathcal{B}} (x^2+y^2)\rho(x,y,z)\,dV & \mathcal{I}_{yz} &= -\int_{\mathcal{B}} yz\rho(x,y,z)\,dV. \end{aligned}$$

mass density function $\rho(x, y, z)$

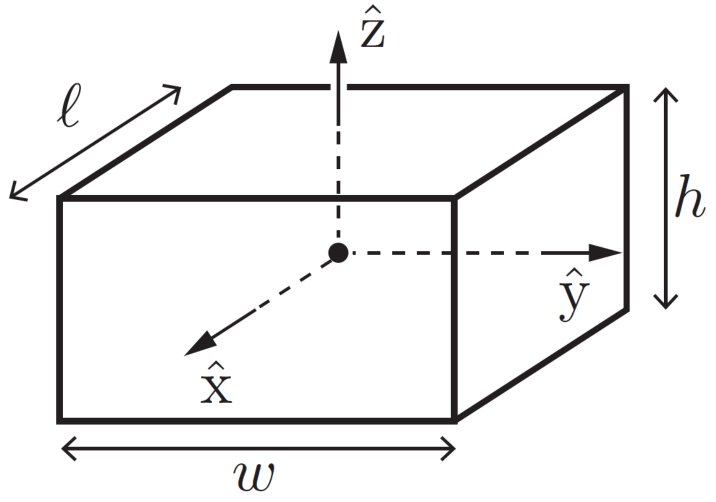
Inertia Matrix

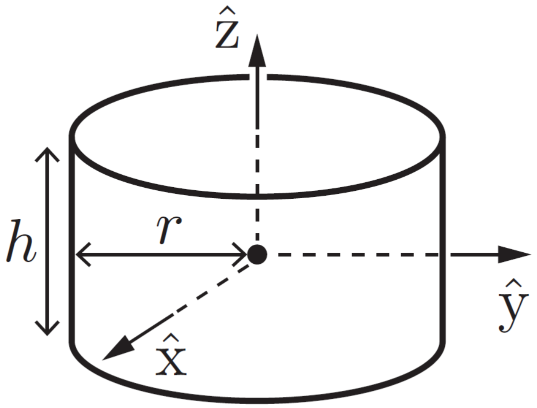
- Principal axes of inertia: eigenvectors and eigenvalues of $\mathcal{I}_b$
  - Directions given by eigenvectors
  - Eigenvalues are principal moments of inertia

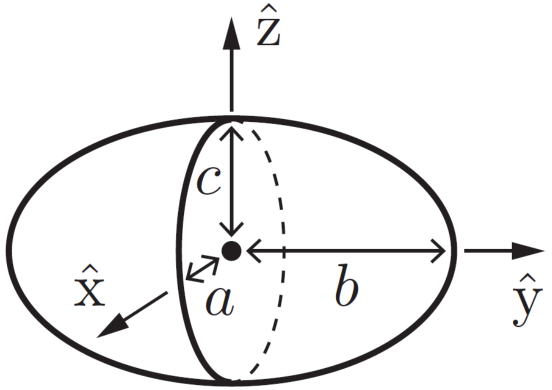
- If the principal axes are aligned with the axes of {b}, $\mathcal{I}_b$ is a diagonal matrix

rotational dynamics $$m_b = \begin{bmatrix} \mathcal{I}_{xx}\dot{\omega}_x + (\mathcal{I}_{zz} - \mathcal{I}_{yy})\omega_y\omega_z \\ \mathcal{I}_{yy}\dot{\omega}_y + (\mathcal{I}_{xx} - \mathcal{I}_{zz})\omega_x\omega_z \\ \mathcal{I}_{zz}\dot{\omega}_z + (\mathcal{I}_{yy} - \mathcal{I}_{xx})\omega_x\omega_y \end{bmatrix}$$ $$\omega_b = (\omega_x, \omega_y, \omega_z)$$

Inertia Matrix







rectangular parallelepiped:

volume $= abc$,

$$\mathcal{I}_{xx} = \mathfrak{m}(w^2 + h^2)/12,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(\ell^2 + h^2)/12,$$
$$\mathcal{I}_{zz} = \mathfrak{m}(\ell^2 + w^2)/12$$

circular cylinder:

volume $= \pi r^2 h$,

$$\mathcal{I}_{xx} = \mathfrak{m}(3r^2 + h^2)/12,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(3r^2 + h^2)/12,$$
$$\mathcal{I}_{zz} = \mathfrak{m}r^2/2$$

ellipsoid:

volume $= 4\pi abc/3$,

$$\mathcal{I}_{xx} = \mathfrak{m}(b^2 + c^2)/5,$$
$$\mathcal{I}_{yy} = \mathfrak{m}(a^2 + c^2)/5,$$
$$\mathcal{I}_{zz} = \mathfrak{m}(a^2 + b^2)/5$$

Inertia Matrix

- Inertia matrix in a rotated frame {c}
- Kinetic energy is the same in different frame

$$
\begin{aligned}
\frac{1}{2}\omega_c^{\mathrm{T}}\mathcal{I}_c\omega_c &= \frac{1}{2}\omega_b^{\mathrm{T}}\mathcal{I}_b\omega_b \\
&= \frac{1}{2}(R_{bc}\omega_c)^{\mathrm{T}}\mathcal{I}_b(R_{bc}\omega_c) \\
&= \frac{1}{2}\omega_c^{\mathrm{T}}(R_{bc}^{\mathrm{T}}\mathcal{I}_b R_{bc})\omega_c.
\end{aligned}
$$

$$\mathcal{I}_c = R_{bc}^{\mathrm{T}}\mathcal{I}_b R_{bc}$$

Steiner's theorem

- The inertia matrix $\mathcal{I}_q$ about a frame aligned with {b}, but at a point in {b} $q = (q_x, q_y, q_z)$, is related to the inertia matrix calculated at the center of mass by

$$\mathcal{I}_q = \mathcal{I}_b + \mathfrak{m}(q^{\mathrm{T}} q I - q q^{\mathrm{T}})$$

- Parallel-axis theorem: the scalar inertia $\mathcal{I}_d$ about an axis parallel to, but a distance d from, an axis through the center of mass is

$$\mathcal{I}_d = \mathcal{I}_{\mathrm{cm}} + \mathfrak{m} d^2$$

Inertia Matrix

- Change of reference frame

$$\mathcal{I}_c = R_{bc}^{\mathrm{T}} \mathcal{I}_b R_{bc}$$

$$\mathcal{I}_q = \mathcal{I}_b + \mathfrak{m}(q^{\mathrm{T}} q I - q q^{\mathrm{T}})$$

Twist-Wrench Formulation

- Linear dynamics $f_b = \mathfrak{m}(\dot{v}_b + [\omega_b]v_b)$
- Rotation dynamics $m_b = \mathcal{I}_b\dot{\omega}_b + [\omega_b]\mathcal{I}_b\omega_b$

$$\begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [\omega_b] & 0 \\ 0 & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$[v]v = v \times v = 0 \text{ and } [v]^{\mathrm{T}} = -[v]$$

$$\begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [\omega_b] & [v_b] \\ 0 & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} - \begin{bmatrix} [\omega_b] & 0 \\ [v_b] & [\omega_b] \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

Twist-Wrench Formulation

Body twist $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$

Body wrench $\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix}$

spatial inertia matrix $\mathcal{G}_b \in \mathbb{R}^{6\times 6}$ $\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix}$

kinetic energy $= \frac{1}{2}\omega_b^{\mathrm{T}}\mathcal{I}_b\omega_b + \frac{1}{2}\mathfrak{m}v_b^{\mathrm{T}}v_b = \frac{1}{2}\mathcal{V}_b^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b$

spatial momentum $\mathcal{P}_b \in \mathbb{R}^6$ $\mathcal{P}_b = \begin{bmatrix} \mathcal{I}_b\omega_b \\ \mathfrak{m}v_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix}\begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \mathcal{G}_b\mathcal{V}_b$

Twist-Wrench Formulation

- Lie bracket of two twists $\mathcal{V}_1 = (\omega_1, v_1)$ and $\mathcal{V}_2 = (\omega_2, v_2)$

$$\left[\begin{array}{cc} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{array}\right]\left[\begin{array}{c} \omega_2 \\ v_2 \end{array}\right] = [\mathrm{ad}_{\mathcal{V}_1}]\mathcal{V}_2 = \mathrm{ad}_{\mathcal{V}_1}(\mathcal{V}_2) \in \mathbb{R}^6$$

$$[\mathrm{ad}_{\mathcal{V}}] = \left[\begin{array}{cc} [\omega] & 0 \\ [v] & [\omega] \end{array}\right] \in \mathbb{R}^{6\times 6}$$

Twist-Wrench Formulation

- Dynamic equations for a single rigid body

$$\begin{aligned}\mathcal{F}_b &= \mathcal{G}_b\dot{\mathcal{V}}_b - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_b}(\mathcal{P}_b) \\ &= \mathcal{G}_b\dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b\end{aligned}$$

- Moment equation for a rigid body

$$m_b = \mathcal{I}_b\dot{\omega}_b - [\omega_b]^{\mathrm{T}}\mathcal{I}_b\omega_b$$

Dynamics in Other Frames

- The kinetic energy is independent of frames

$$\begin{aligned}\frac{1}{2}\mathcal{V}_a^{\mathrm{T}}\mathcal{G}_a\mathcal{V}_a &= \frac{1}{2}\mathcal{V}_b^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b \\ &= \frac{1}{2}([\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a)^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a \\ &= \frac{1}{2}\mathcal{V}_a^{\mathrm{T}}\underbrace{[\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]}_{\mathcal{G}_a}\mathcal{V}_a;\end{aligned}$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- The spatial inertia matrix $\mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]$
- Equations of motion in frame {a}

$$\mathcal{F}_a = \mathcal{G}_a\dot{\mathcal{V}}_a - [\mathrm{ad}_{\mathcal{V}_a}]^{\mathrm{T}}\mathcal{G}_a\mathcal{V}_a$$

Summary

- Dynamics of a single rigid body
  - Linear dynamics

  - Rotational dynamics

$$\begin{aligned}\mathcal{F}_b &= \mathcal{G}_b \dot{\mathcal{V}}_b - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_b}(\mathcal{P}_b) \\ &= \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}} \mathcal{G}_b \mathcal{V}_b\end{aligned}$$

Further Reading

- Chapter 8 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.