Dynamics of Open Chains: Lagrangian Formulation

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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Robot Dynamics

- Study motion of robots with the forces and torques that cause them

- Equations of motion
  - A set of second-order differential equations

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

Joint variables $\theta \in \mathbb{R}^n$

Joint forces and torques $\tau \in \mathbb{R}^n$

$M(\theta) \in \mathbb{R}^{n \times n}$ a symmetric positive-definite **mass matrix**

$h(\theta, \dot{\theta}) \in \mathbb{R}^n$ forces that lump together centripetal, Coriolis, gravity, and friction terms that depend on θ and $\dot{\theta}$

Forward and Inverse Dynamics

- Forward dynamics
  - Given robot state $(\theta, \dot{\theta})$ and the joint forces and torques
  - Determine the robot's acceleration

$$\ddot{\theta} = M^{-1}(\theta)\left(\tau - h(\theta, \dot{\theta})\right)$$

Simulation

- Inverse dynamics
  - Given robot state $(\theta, \dot{\theta})$ and a desired acceleration
  - Find the joint forces and torques

Control

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

Robot Dynamics

- Lagrangian formulation
  - Kinetic energy and potential energy
- Newton-Euler formulation
  - $F = ma$

Lagrangian Formulation

- Generalized coordinates
  - Choose a set of independent coordinates $q \in \mathbb{R}^n$ that describes the system's configuration
- Generalized forces $f \in \mathbb{R}^n$
  - Power $f^{\mathrm{T}}\dot{q}$
- Lagrangian function $\mathcal{L}(q, \dot{q}) = \mathcal{K}(q, \dot{q}) - \mathcal{P}(q)$

  Kinetic energy Potential energy

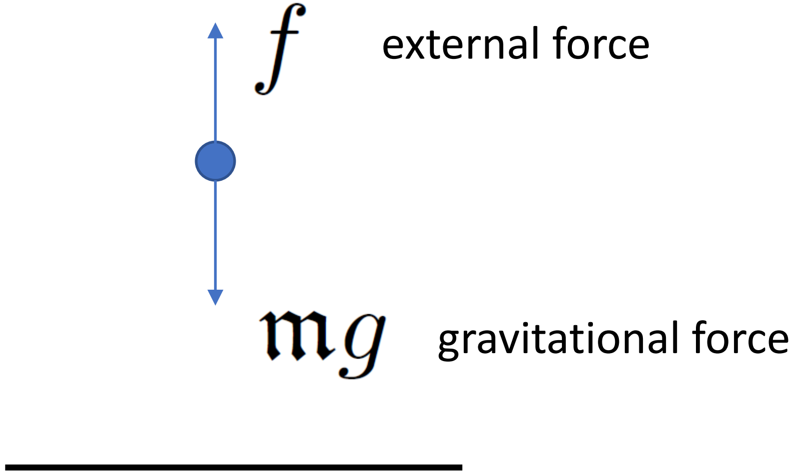
- Equations of motion

$$f = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q}$$

Euler-Lagrange equations with external forces

Lagrangian Formulation

- Example: a particle of mass m constrained to move on a vertical line



Generalized coordinate: height of the particle $x \in \mathbb{R}$

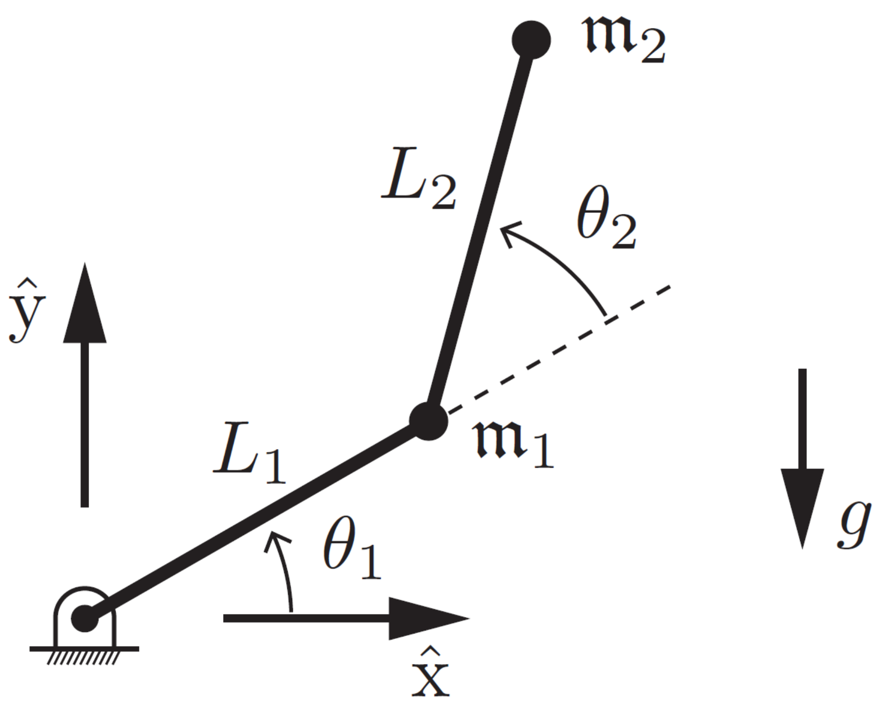
Newton's second law $f - \mathfrak{m}g = \mathfrak{m}\ddot{x}$

Lagrangian formulation

$$\mathcal{L}(x, \dot{x}) = \mathcal{K}(x, \dot{x}) - \mathcal{P}(x) = \frac{1}{2}\mathfrak{m}\dot{x}^2 - \mathfrak{m}gx$$

$$f = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \mathfrak{m}\ddot{x} + \mathfrak{m}g$$

Lagrangian Formulation



A 2R open chain under gravity

Two links with point masses

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 \\ L_1 \sin\theta_1 \end{bmatrix},$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 \\ L_1 \cos\theta_1 \end{bmatrix} \dot{\theta}_1$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix},$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

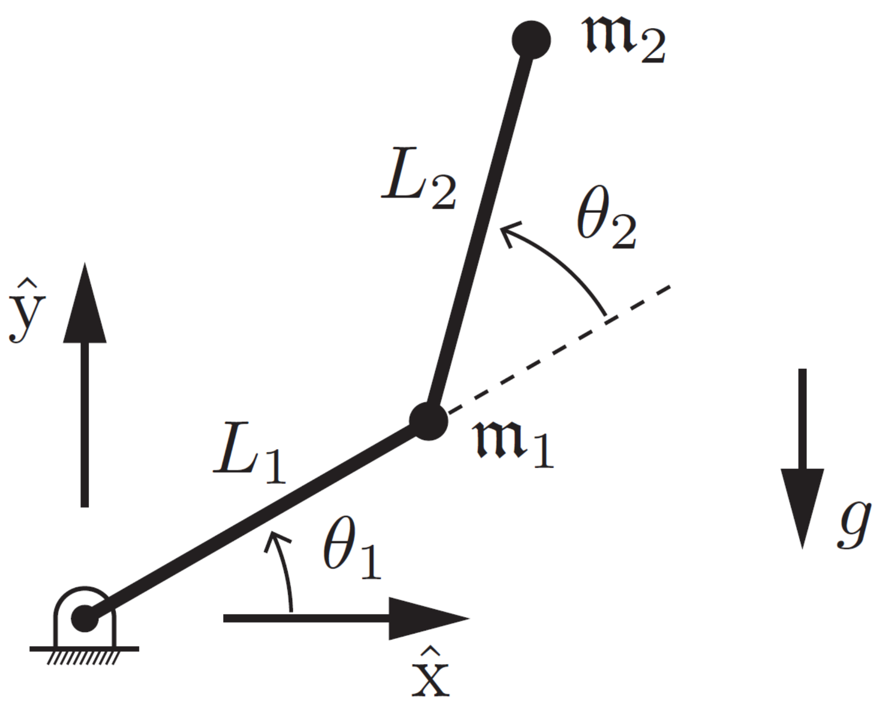
Generalized coordinate $\theta = (\theta_1, \theta_2)$

Generalized forces $\tau = (\tau_1, \tau_2)$ Joint torques

Lagrangian

$$\mathcal{L}(\theta, \dot{\theta}) = \sum_{i=1}^{2} (\mathcal{K}_i - \mathcal{P}_i)$$

Lagrangian Formulation



A 2R open chain under gravity

Lagrangian $\mathcal{L}(\theta,\dot\theta) = \sum_{i=1}^{2}(\mathcal{K}_i - \mathcal{P}_i)$

$$\mathcal{K}_1 = \frac{1}{2}\mathfrak{m}_1(\dot x_1^2 + \dot y_1^2) = \frac{1}{2}\mathfrak{m}_1 L_1^2\dot\theta_1^2$$

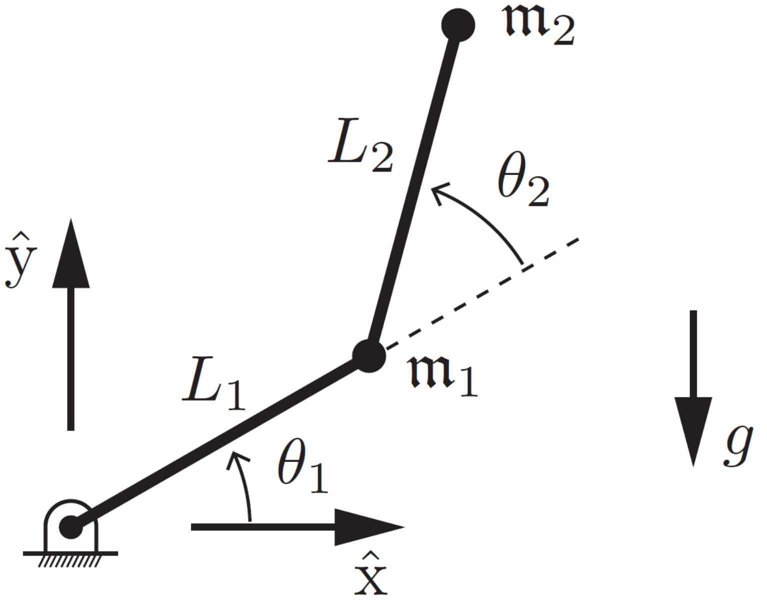
$$\begin{aligned}\mathcal{K}_2 &= \frac{1}{2}\mathfrak{m}_2(\dot x_2^2 + \dot y_2^2)\\ &= \frac{1}{2}\mathfrak{m}_2\left((L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\dot\theta_1^2 + 2(L_2^2 + L_1L_2\cos\theta_2)\dot\theta_1\dot\theta_2 + L_2^2\dot\theta_2^2\right)\end{aligned}$$

$$\begin{aligned}\mathcal{P}_1 &= \mathfrak{m}_1 g y_1 = \mathfrak{m}_1 g L_1 \sin\theta_1,\\ \mathcal{P}_2 &= \mathfrak{m}_2 g y_2 = \mathfrak{m}_2 g(L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2))\end{aligned}$$

$$\tau_i = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta_i} - \frac{\partial\mathcal{L}}{\partial\theta_i}, \qquad i = 1,2.$$

$$\begin{aligned}\tau_1 &= \left(\mathfrak{m}_1L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\right)\ddot\theta_1\\ &\quad + \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot\theta_2 - \mathfrak{m}_2L_1L_2\sin\theta_2(2\dot\theta_1\dot\theta_2 + \dot\theta_2^2)\\ &\quad + (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1+\theta_2),\\ \tau_2 &= \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot\theta_1 + \mathfrak{m}_2L_2^2\ddot\theta_2 + \mathfrak{m}_2L_1L_2\dot\theta_1^2\sin\theta_2\\ &\quad + \mathfrak{m}_2 g L_2\cos(\theta_1+\theta_2).\end{aligned}$$

Lagrangian Formulation



A 2R open chain under gravity

Mass matrix | Product-velocity term | Gravity term

$$\tau = M(\theta)\ddot{\theta} + \underbrace{c(\theta,\dot{\theta}) + g(\theta)}_{h(\theta,\dot{\theta})},$$

$$M(\theta) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix}$$

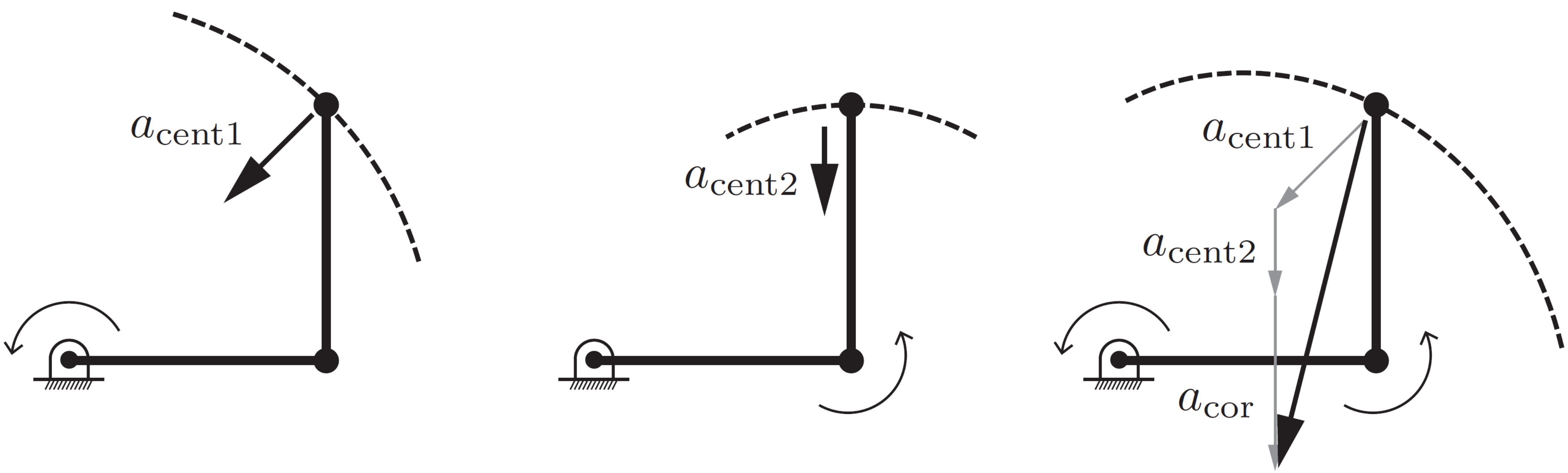
$$c(\theta,\dot{\theta}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix}$$

Coriolis and centripetal torques

$$g(\theta) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1+\theta_2) \\ \mathfrak{m}_2 g L_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

gravitational torques

Lagrangian Formulation



Assuming $\ddot{\theta} = 0$

$$\begin{bmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -L_1\dot{\theta}_1^2 \\ -L_2\dot{\theta}_1^2 - L_2\dot{\theta}_2^2 \end{bmatrix}}_{\text{centripetal terms}} + \underbrace{\begin{bmatrix} 0 \\ -2L_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}}_{\text{Coriolis terms}}$$

Lagrangian Formulation

- For general n-link open chain and all joints are actuated
- Generalized coordinates are joint values $\theta \in \mathbb{R}^n$
- Generalized forces $\tau \in \mathbb{R}^n$
  - Revolute joint -> torque
  - Prismatic joint -> force
- Lagrangian $\mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta)$

$$\mathcal{K}(\theta) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j = \frac{1}{2}\dot{\theta}^{\mathrm{T}} M(\theta)\dot{\theta}$$

$n \times n$ mass matrix $M(\theta)$

Lagrangian Formulation

- Dynamic equations $$\tau_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i}, \qquad i = 1, \ldots, n.$$

$$\tau_i = \sum_{j=1}^{n} m_{ij}(\theta)\ddot{\theta}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_j\dot{\theta}_k + \frac{\partial \mathcal{P}}{\partial \theta_i}, \qquad i = 1, \ldots, n,$$

Christoffel symbols of the first kind $$\Gamma_{ijk}(\theta) = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial \theta_k} + \frac{\partial m_{ik}}{\partial \theta_j} - \frac{\partial m_{jk}}{\partial \theta_i}\right)$$

$$\tau = M(\theta)\ddot{\theta} + \dot{\theta}^{\mathrm{T}}\Gamma(\theta)\dot{\theta} + g(\theta)$$

$\Gamma(\theta)$ $n \times n \times n$

$g(\theta) = \partial \mathcal{P}/\partial \theta$

$$\dot{\theta}^{\mathrm{T}}\Gamma(\theta)\dot{\theta} = \begin{bmatrix} \dot{\theta}^{\mathrm{T}}\Gamma_1(\theta)\dot{\theta} \\ \dot{\theta}^{\mathrm{T}}\Gamma_2(\theta)\dot{\theta} \\ \vdots \\ \dot{\theta}^{\mathrm{T}}\Gamma_n(\theta)\dot{\theta} \end{bmatrix}$$

$\Gamma_i(\theta)$ is an $n \times n$ matrix with (j,k)th entry Γ_{ijk}

Lagrangian Formulation

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta)$$

Coriolis matrix $C(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$

$$c_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_k$$

Understanding the Mass Matrix

- Kinetic energy $\mathcal{K}(\theta) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j = \frac{1}{2}\dot{\theta}^{\mathrm{T}} M(\theta)\dot{\theta}$
- A generalization of point mass $\frac{1}{2} m v^{\mathrm{T}} v$
- M is positive definite $\dot{\theta}^{\mathrm{T}} M(\theta)\dot{\theta} > 0$ for all $\dot{\theta} \neq 0$
  - Mass of a point mass is always positive

Understanding the Mass Matrix

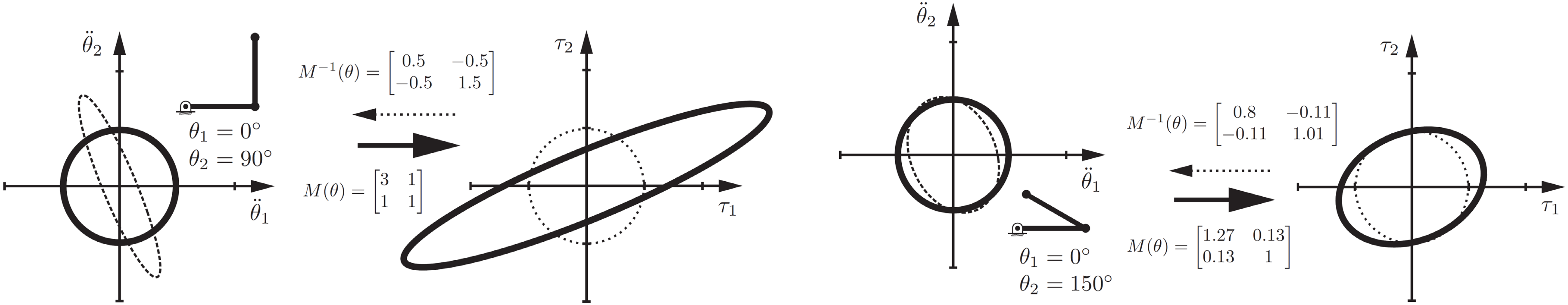
- For a point mass $f = m\ddot{x}$
  - m is independent of the acceleration direction

- A mass matrix presents different effective mass in different acceleration directions

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta)$$

Understanding the Mass Matrix

$$\dot{\theta} = 0$$



$$\{\ddot{\theta} \mid \ddot{\theta}^{\mathrm{T}}\ddot{\theta} = 1\}$$

$$M(\theta) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix},$$

$$L_1 = L_2 = \mathfrak{m}_1 = \mathfrak{m}_2 = 1$$

Summary

- Robot Dynamics

- Lagrangian formulation

Further Reading

- Chapter 8 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.