Forward Kinematics and Denavit-Hartenberg Parameters

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation
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Forward Kinematics

• Forward kinematics of a robot: calculation of the position and orientation of its end-effector from its joint coordinates $\theta$

• Recall robot links and joints
Forward Kinematics

- End-effector frame \{4\}
- Joint angles \((\theta_1, \theta_2, \theta_3)\)
- Position and orientation of the end-effector frame

\[
\begin{align*}
x &= L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3), \\
y &= L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3), \\
\phi &= \theta_1 + \theta_2 + \theta_3.
\end{align*}
\]

Forward kinematics of a 3R planar open chain.
Forward Kinematics

- General cases
  - Attaching frames to links
  - Using homogeneous transformations

\[
T_{04} = T_{01} T_{12} T_{23} T_{34}
\]

- Depend only on the joint variable \( \theta_i \)

Forward kinematics of a 3R planar open chain.
Forward Kinematics

- A different approach
- Define $M$ to the position and orientation of frame $\{4\}$ when all the joint angles are zeros ("home" or "zero" position of the robot)

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Consider each revolute joint as a zero-pitch screw-axis expressed in the $\{0\}$ frame

Linear velocity of the origin of $\{0\}$ in the $\{0\}$ frame

$$v_3 = -\omega_3 \times q_3$$

$$q_3 = (L_1 + L_2, 0, 0)$$

Forward kinematics of a 3R planar open chain.

For joint 3

$$s_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$
Forward Kinematics

\[
[S_3] = \begin{bmatrix} \omega & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
T_{04} = e^{[S_3] \theta_3} M \quad \text{(for } \theta_1 = \theta_2 = 0)\]

\[
T_{04} = e^{[S_2] \theta_2} e^{[S_3] \theta_3} M \quad \text{(for } \theta_1 = 0)\]

\[
[S_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
[S_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

a product of matrix exponentials
(does not use any frame references, only \{0\} and M)
Forward Kinematics

- Method 1: uses homogeneous transformations
  - Need to define the coordinates of frames

- Method 2: uses screw-axis representations of transformations
  - No need to define frame references
Denavit-Hartenberg Parameters

- Attach reference frames to each link of an open chain
- Derive forward kinematics using the relative displacements between adjacent line frames
- For a chain with $n$ 1DOF joints, 0,...,n
  - The ground link is 0
  - The end-effect frame is attached to link $n$

$$T_{0n}(\theta_1, \ldots, \theta_n) = T_{01}(\theta_1)T_{12}(\theta_2) \cdots T_{n-1,n}(\theta_n)$$

$$T_{i,i-1} \in SE(3)$$
Denavit-Hartenberg Parameters

- Assigning link frames
  - \( \hat{z}_i \)-axis coincides with joint axis \( i \)
  - \( \hat{z}_{i-1} \)-axis coincides with joint axis \( i-1 \)
  - Origin of the link frame
    - Find the line segment that orthogonally intersects both the joint axes
    - Origin of frame \( \{i-1\} \) is the intersection of the line and the joint axis \( i-1 \) \( \hat{z}_{i-1} \) and \( \hat{z}_i \)
  - \( \hat{x} \)-axis in the direction of the mutual perpendicular line pointing from \( (i-1) \)-axis to \( i \)-axis
  - \( \hat{y} \)-axis given by \( \hat{x} \times \hat{y} = \hat{z} \)
Denavit-Hartenberg (D-H) Parameters

- **Link length**: the length of the mutual perpendicular line $a_{i-1}$
  - Not the actual length of the physical link

- **Line twist** $\alpha_{i-1}$
  - the angle from $\hat{z}_{i-1}$ to $\hat{z}_i$, measured about $\hat{x}_{i-1}$

- **Line offset** $d_i$
  - Distance from the intersection to the origin of the link-$i$ frame

- **Joint angle** $\phi_i$
  - the angle from $\hat{x}_{i-1}$ to $\hat{x}_i$, measured about the $\hat{z}_i$-axis
D-H Parameters

• For an open chain with n 1DOF joints, 4n D-H parameters

• For an open chain with all joints revolute
  - Link lengths $a_{i-1}$
  - Line twists $\alpha_{i-1}$
  - Line offsets $d_i$
  - Joint angle parameters are the joint variables
D-H Parameters

- When adjacent revolute joint axes intersect
  - No mutual perpendicular line
  - Link length 0
  - \( \hat{x}_{i-1} \) perpendicular to the plane spanned by \( \hat{z}_{i-1} \) and \( \hat{z}_i \)

- When adjacent revolute joint axes are parallel
  - Many possibilities for a mutually perpendicular line
  - Choose the one that is most physically intuitive and results in many zero parameters as possible
D-H Parameters

- Prismatic joints

\[ \begin{align*}
  \hat{z}_i - \text{axis} & \quad \text{positive direction of translation} \\
  d_i & \quad \text{link offset is the joint variable} \\
  \phi_i & \quad \text{joint angle is constant} \\
  \hat{x}_i - \text{axis} & \quad \text{in the direction of the mutual perpendicular line pointing from (i-1)-axis to i-axis} \\
  \hat{y}_i - \text{axis} & \quad \text{given by } \hat{x} \times \hat{y} = \hat{z}
\end{align*} \]
Forward Kinematics with D-H Parameters

- Link frame transformation

\[
T_{i-1,i} = \text{Rot}(\hat{x}, \alpha_{i-1})\text{Trans}(\hat{x}, a_{i-1})\text{Trans}(\hat{z}, d_i)\text{Rot}(\hat{z}, \phi_i)
\]

\[
= \begin{bmatrix}
\cos \phi_i & -\sin \phi_i & 0 & a_{i-1} \\
\sin \phi_i \cos \alpha_{i-1} & \cos \phi_i \cos \alpha_{i-1} & -\sin \alpha_{i-1} & -d_i \sin \alpha_{i-1} \\
\sin \phi_i \sin \alpha_{i-1} & \cos \phi_i \sin \alpha_{i-1} & \cos \alpha_{i-1} & d_i \cos \alpha_{i-1} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\text{Rot}(\hat{z}, \phi_i) = \begin{bmatrix}
\cos \phi_i_{-1} & -\sin \phi_i_{-1} & 0 & 0 \\
\sin \phi_i_{-1} & \cos \phi_i_{-1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\text{Trans}(\hat{z}, d_i) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\text{Trans}(\hat{x}, a_{i-1}) = \begin{bmatrix}
1 & 0 & 0 & a_{i-1} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\text{Rot}(\hat{x}, \alpha_{i-1}) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \alpha_{i-1} & -\sin \alpha_{i-1} & 0 \\
0 & \sin \alpha_{i-1} & \cos \alpha_{i-1} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Forward Kinematics with D-H Parameters

A 3R spatial open chain in its zero position

D-H Parameters

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\phi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>$90^\circ$</td>
<td>$L_1$</td>
<td>0</td>
<td>$\theta_2 - 90^\circ$</td>
</tr>
<tr>
<td>3</td>
<td>$-90^\circ$</td>
<td>$L_2$</td>
<td>0</td>
<td>$\theta_3$</td>
</tr>
</tbody>
</table>
Forward Kinematics with D-H Parameters

An RRRP spatial open chain in its zero position

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\phi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>$90^\circ$</td>
<td>0</td>
<td>0</td>
<td>$\theta_2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$L_2$</td>
<td>0</td>
<td>$\theta_3 + 90^\circ$</td>
</tr>
<tr>
<td>4</td>
<td>$90^\circ$</td>
<td>0</td>
<td>$\theta_4$</td>
<td>0</td>
</tr>
</tbody>
</table>
Forward Kinematics with D-H Parameters

A 6R spatial open chain in its zero position

<table>
<thead>
<tr>
<th>i</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\phi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>90°</td>
<td>0</td>
<td>0</td>
<td>$\theta_2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$L_1$</td>
<td>0</td>
<td>$\theta_3 + 90°$</td>
</tr>
<tr>
<td>4</td>
<td>90°</td>
<td>0</td>
<td>$L_2$</td>
<td>$\theta_4 + 180°$</td>
</tr>
<tr>
<td>5</td>
<td>90°</td>
<td>0</td>
<td>0</td>
<td>$\theta_5 + 180°$</td>
</tr>
<tr>
<td>6</td>
<td>90°</td>
<td>0</td>
<td>0</td>
<td>$\theta_6$</td>
</tr>
</tbody>
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Summary

• Forward kinematics

• Denavit-Hartenberg Parameters
Further Reading
