Exponential Coordinates of Rigid-Body Motions and Wrenches

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

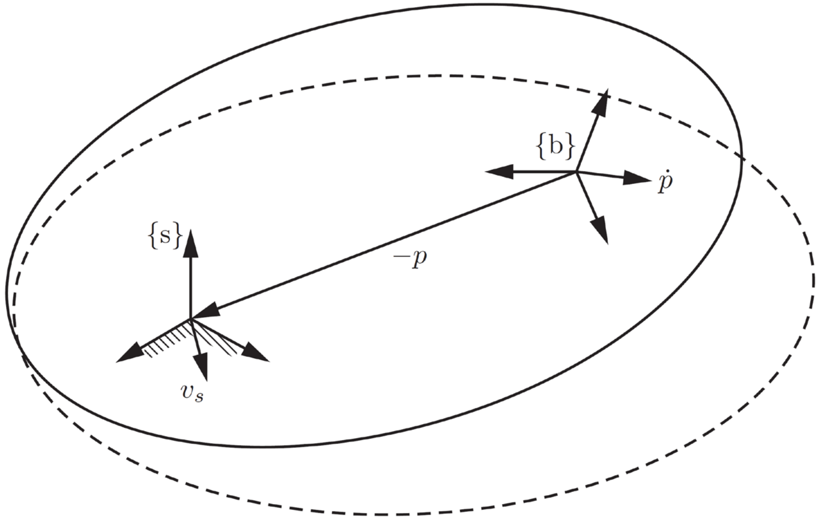
Professor Yu Xiang

The University of Texas at Dallas

Twists and Screw Axes

- Twist

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6 \quad \mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6$$



- A screw axis is a normalized twist

$$\mathcal{S}\dot{\theta} = \mathcal{V} \qquad \mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

Exponential Coordinates of Rigid-Body Motions

- **Chasles-Mozzi theorem**: every rigid-body displacement can be expressed as displacement along a fixed screw axis S in space

- Exponential coordinates of a homogeneous transformation T

$$\mathcal{S}\theta \in \mathbb{R}^6$$

Screw axis

Distance along the screw axis

$$\mathcal{S} = (\omega, v) \qquad \|\omega\| = 1$$

θ Angle of rotation

$$\omega = 0 \quad \|v\| = 1$$

θ Linear distance along the axis

Exponential Coordinates of Rigid-Body Motions

- Exponential coordinates of a homogeneous transformation T

$$\exp : \quad [\mathcal{S}]\theta \in se(3) \quad \rightarrow \quad T \in SE(3)$$

$$\log : \quad T \in SE(3) \quad \rightarrow \quad [\mathcal{S}]\theta \in se(3)$$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

Matrix Exponential

$$e^{[\mathcal{S}]\theta} = I + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \cdots$$

$$= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

$$G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \cdots$$

$$[\omega]^3 = -[\omega]$$

$$= I\theta + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\omega] + \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \cdots\right)[\omega]^2$$

$$= I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2$$

Matrix Exponential

$$\mathcal{S} = (\omega, v) \qquad \theta \in \mathbb{R}$$

If $\|\omega\| = 1$

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}$$

If $\omega = 0$ and $\|v\| = 1$

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$$

Matrix Logarithm

- Given $(R, p) \in SE(3)$, one can find $\mathcal{S} = (\omega, v)$ and θ

$$e^{[\mathcal{S}]\theta} = \left[\begin{array}{cc} R & p \\ 0 & 1 \end{array} \right]$$

- Matrix Logarithm of T = (R, p)

$$[\mathcal{S}]\theta = \left[\begin{array}{cc} [\omega]\theta & v\theta \\ 0 & 0 \end{array} \right] \in se(3)$$

Matrix Logarithm Algorithm

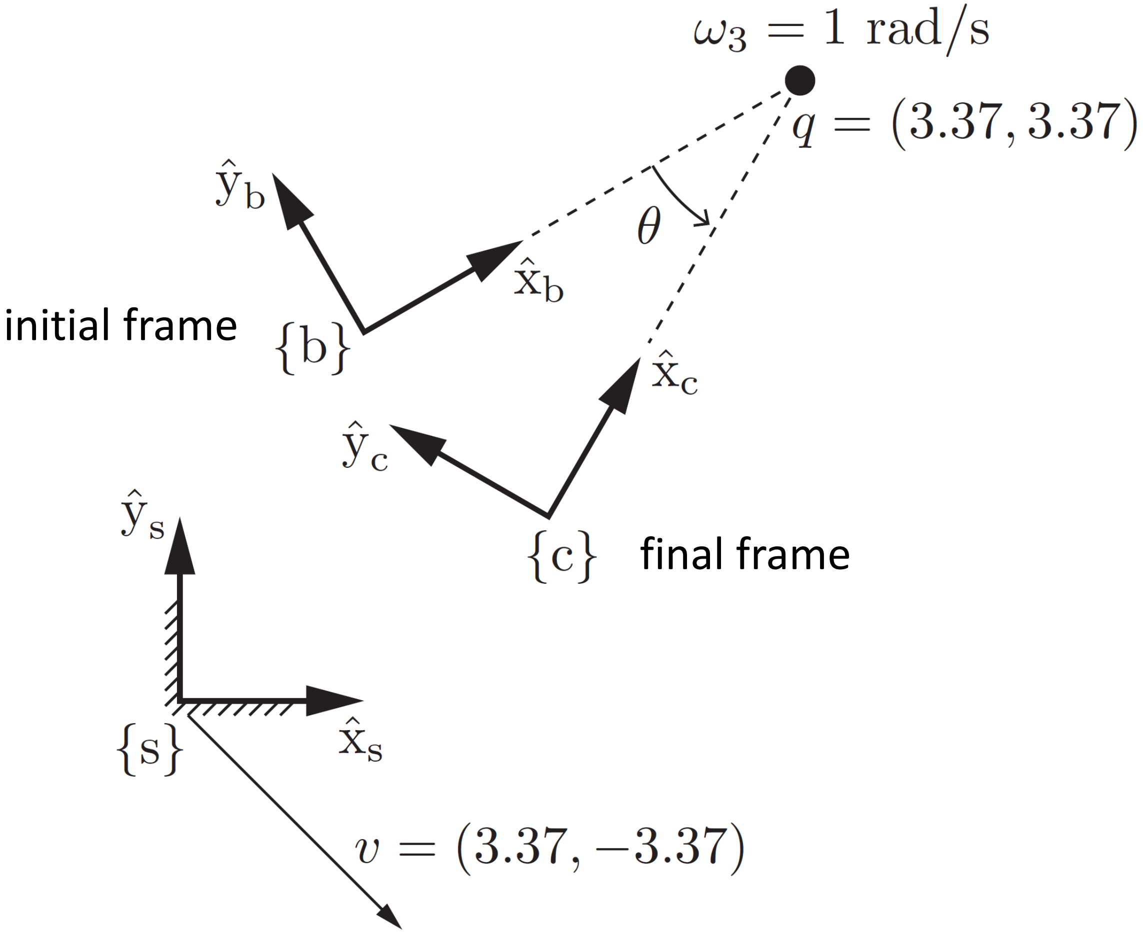
- Given $(R, p) \in SE(3)$, how to find $\mathcal{S} = (\omega, v)$ and θ ?
  - If $R = I$ then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$
  - Otherwise, use the matrix logarithm on SO(3) to determine ω, θ for R

$$v = G^{-1}(\theta)p$$

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2$$

Exercise

Matrix Exponential and Matrix Logarithm



$$T_{sb} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 1 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

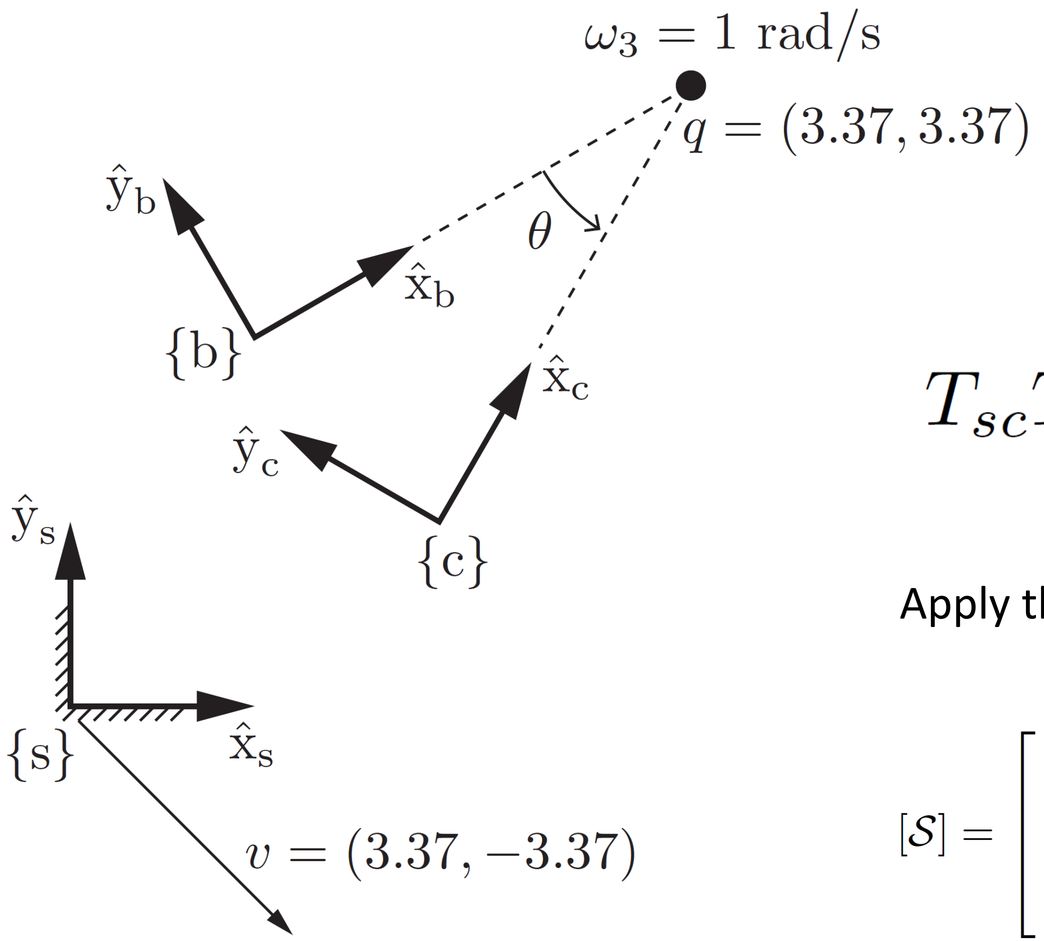
$$T_{sc} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 2 \\ \sin 60^\circ & \cos 60^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

motion occurs in the $\hat{x}_s$–$\hat{y}_s$-plane

Screw axis: $\hat{z}_s$-axis Zero pitch

$$\mathcal{S} = (\omega, v) \qquad \begin{aligned} \omega &= (0, 0, \omega_3), \\ v &= (v_1, v_2, 0). \end{aligned}$$

Matrix Exponential and Matrix Logarithm



Seek screw motion to displace {b} to {c}

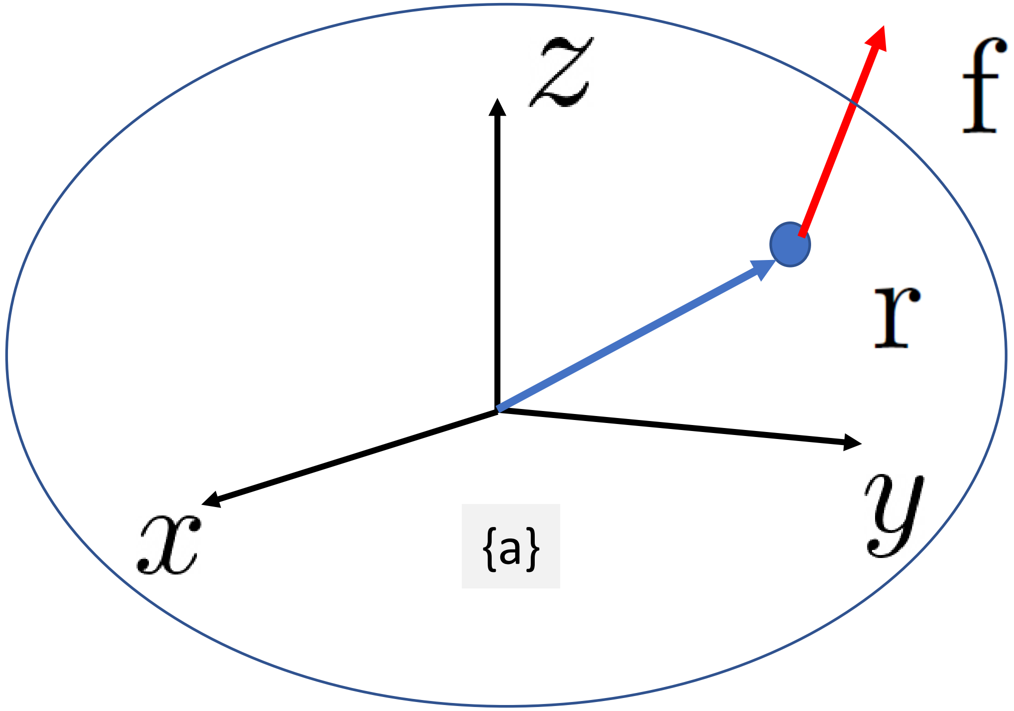
$$T_{sc} = e^{[\mathcal{S}]\theta} T_{sb}$$

$$T_{sc}T_{sb}^{-1} = e^{[\mathcal{S}]\theta} \qquad [\mathcal{S}] = \begin{bmatrix} 0 & -\omega_3 & 0 & v_1 \\ \omega_3 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply the matrix logarithm algorithm to $T_{sc}T_{sb}^{-1}$

$$[\mathcal{S}] = \begin{bmatrix} 0 & -1 & 0 & 3.37 \\ 1 & 0 & 0 & -3.37 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3.37 \\ -3.37 \\ 0 \end{bmatrix}, \quad \theta = \frac{\pi}{6} \text{ rad (or } 30^\circ)$$

Torque



A rigid body

Point $r_a \in \mathbb{R}^3$

Force $f_a \in \mathbb{R}^3$

Torque or Moment

$$m_a \in \mathbb{R}^3$$

$$m_a = r_a \times f_a$$

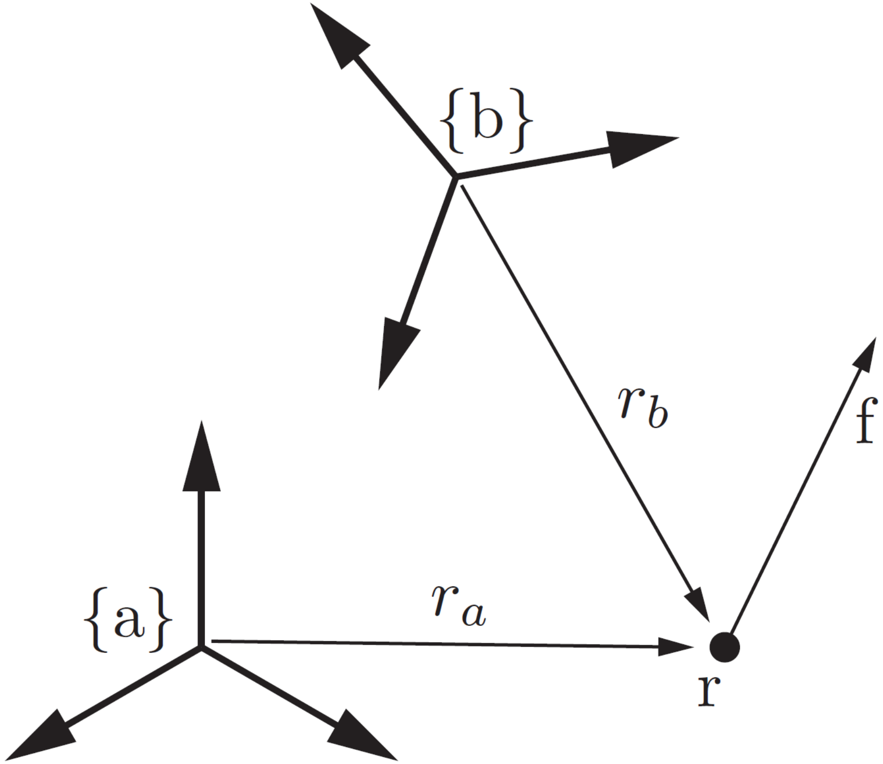
Spatial Force or Wrench

- Merge moment and force in frame {a}

Wrench $$\mathcal{F}_a = \left[\begin{array}{c} m_a \\ f_a \end{array} \right] \in \mathbb{R}^6$$

- If more than one wrenches act on a rigid body, the total wrench is the vector sum of the wrenches
- Pure moment: a wrench with a zero linear component

Wrench in Different Frames



- Power generated by (F, V) are the same

$$\mathcal{V}_b^{\mathrm{T}}\mathcal{F}_b = \mathcal{V}_a^{\mathrm{T}}\mathcal{F}_a$$

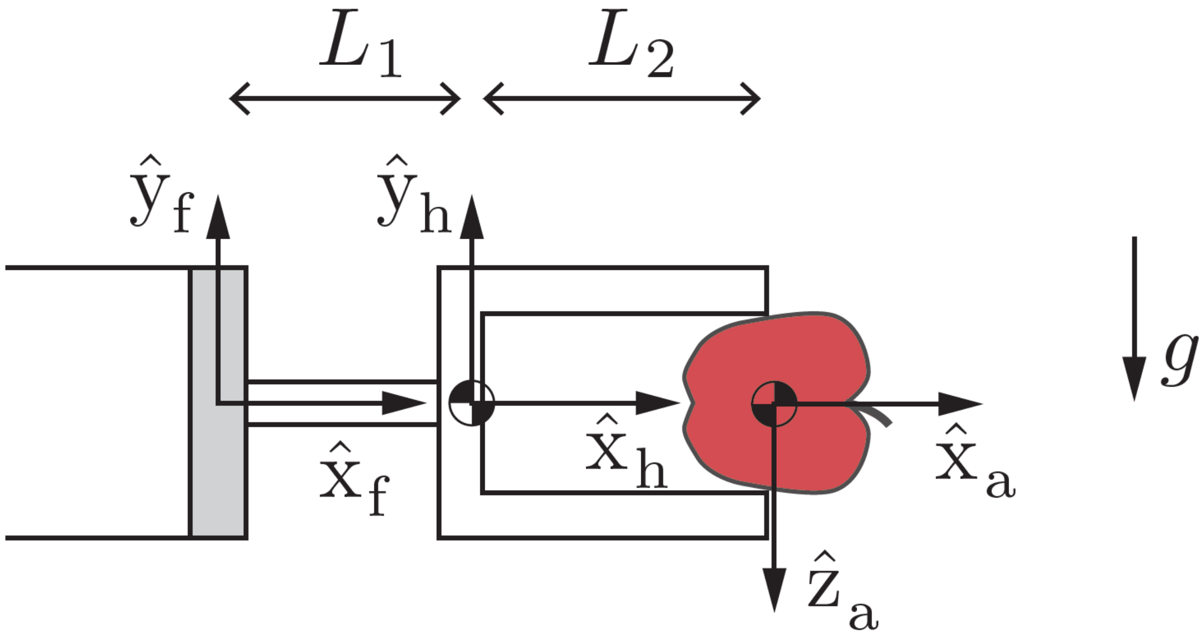
$$\mathcal{V}_a \ = \ [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$$

$$\begin{aligned}\mathcal{V}_b^{\mathrm{T}}\mathcal{F}_b &= ([\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b)^{\mathrm{T}}\mathcal{F}_a \\ &= \mathcal{V}_b^{\mathrm{T}}[\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}}\mathcal{F}_a.\end{aligned}$$

$$\mathcal{F}_b = [\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}}\mathcal{F}_a$$

$$\mathcal{F}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{F}_b$$

Wrench Example



A robot hand holding an apple subject to gravity

- Apple mass 0.1 kg
- Gravity g=10 $\mathrm{m/s^2}$
- Mass of hand 0.5 kg

What is the force and torque measured by the six-axis force-torque sensor between the hand and the robot arm?

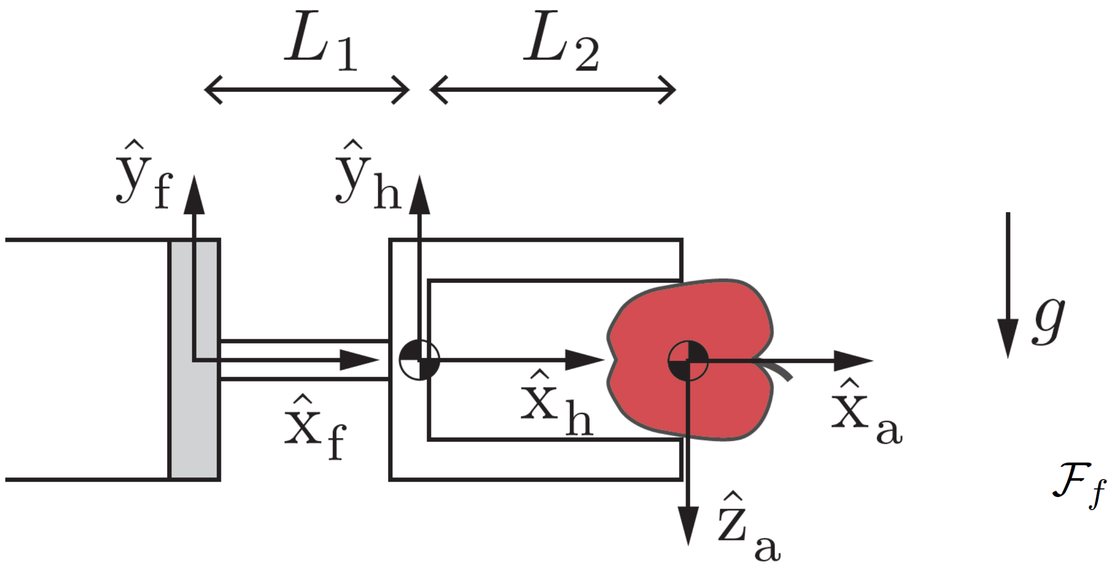
- Frame {f} at the sensor
- Frame {h} at the center of mass of hand
- Frame {a} at the center of mass of apple
- Gravitational wrench on hand in {h}

$$\mathcal{F}_h = (0, 0, 0, 0, -5\ \mathrm{N}, 0)$$

- Gravitational wrench on apple in {a}

$$\mathcal{F}_a = (0, 0, 0, 0, 0, 1\ \mathrm{N})$$

Wrench Example



A robot hand holding an apple subject to gravity

$$L_1 = 10 \text{ cm} \qquad L_2 = 15 \text{ cm}$$

$$T_{hf} = \begin{bmatrix} 1 & 0 & 0 & -0.1 \text{ m} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_{af} = \begin{bmatrix} 1 & 0 & 0 & -0.25 \text{ m} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \mathcal{F}_f &= [\mathrm{Ad}_{T_{hf}}]^{\mathrm{T}} \mathcal{F}_h + [\mathrm{Ad}_{T_{af}}]^{\mathrm{T}} \mathcal{F}_a \\ &= [0\ 0\ -0.5 \text{ Nm}\ 0\ -5 \text{ N}\ 0]^{\mathrm{T}} + [0\ 0\ -0.25 \text{ Nm}\ 0\ -1 \text{ N}\ 0]^{\mathrm{T}} \\ &= [0\ 0\ -0.75 \text{ Nm}\ 0\ -6 \text{ N}\ 0]^{\mathrm{T}}. \end{aligned}$$

Summary

- Exponential Coordinates of Rigid-Body Motions
- Matrix Logarithm of Rigid-Body Motions
- Torques
- Wrenches

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017