Twists and Screw Axes

CS 6301 Special Topics: Introduction to Robot Manipulation and Navigation

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Homogenous Transformation Matrices

- Consider body frame {b} in a fixed frame {s}
  - 3D rotation $R \in SO(3)$
  - 3D position $p \in \mathbb{R}^3$

- Special Euclidean group SE(3) or homogenous transformation matrices

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Twists

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$

- Consider both linear and angular velocities when the rigid-body moves

Recall angular velocity $\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}$

$$\dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

$$\dot{R}R^{-1} = [\omega_s]$$ Angular velocity expressed in {s}

$$R^{-1}\dot{R} = [\omega_b]$$ Angular velocity expressed in {b}

Twists

- Let's compute

$$T^{-1}\dot{T} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} R^{\mathrm{T}}\dot{R} & R^{\mathrm{T}}\dot{p} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}.$$

Recall

$$\dot{R}R^{-1} = [\omega_s]$$
$$R^{-1}\dot{R} = [\omega_b]$$

$$R^{\mathrm{T}}\dot{p} = v_b$$

linear velocity of a point at the origin of {b} expressed in {b}

Twists

- Body twist (spatial velocity in the body frame)

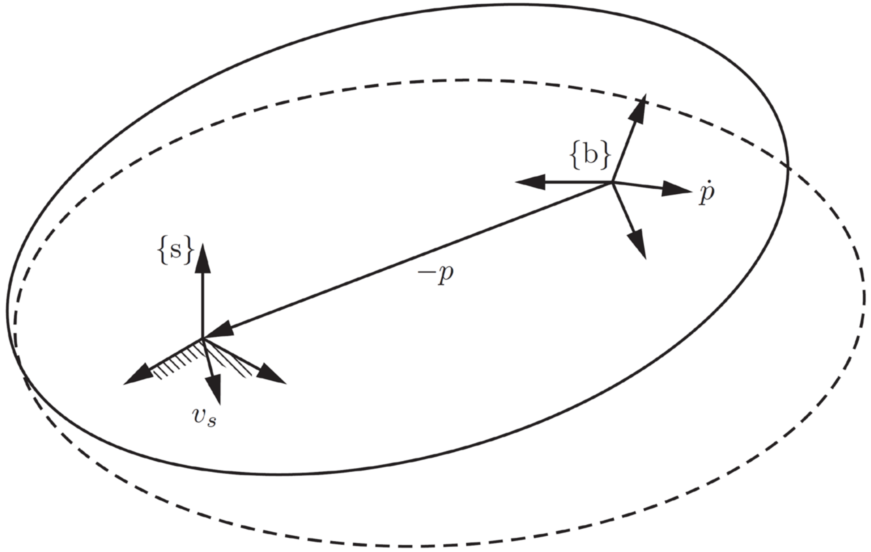
$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$$

matrix representation

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$$

$[\omega_b] \in so(3)$ $\quad v_b \in \mathbb{R}^3$ **linear velocity of a point at the origin of {b} expressed in {b}**

Twists



- Similarly

$$\dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \dot{R}R^{\mathrm{T}} & \dot{p} - \dot{R}R^{\mathrm{T}}p \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}.$$

$$[\omega_s] = \dot{R}R^{\mathrm{T}}$$

$v_s = \dot{p} - \dot{R}R^{\mathrm{T}}p$ Not the linear velocity in fixed frame $\dot{p}$

$$v_s = \dot{p} - \omega_s \times p = \dot{p} + \omega_s \times (-p)$$

Imagining the moving body to be infinitely large

Linear velocity of a point at the origin of {s} expressed in {s}

Twists

- Spatial twist (spatial velocity in the space frame)

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6 \qquad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3)$$

- Relationship

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}[\mathcal{V}_s]\,T \end{aligned}$$

$$[\mathcal{V}_s] = T\,[\mathcal{V}_b]\,T^{-1}$$

Twists

- Relationship between body twist and space twist

$$[\mathcal{V}_s] = \begin{bmatrix} R[\omega_b]R^{\mathrm{T}} & -R[\omega_b]R^{\mathrm{T}}p + Rv_b \\ 0 & 0 \end{bmatrix}$$

$$R[\omega]R^{\mathrm{T}} = [R\omega] \qquad\qquad [\omega]p = -[p]\omega$$

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$6 \times 6$$

Adjoint Representations

- The adjoint representation of $T = (R, p) \in SE(3)$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- The adjoint map associated with T

$$\mathcal{V} \in \mathbb{R}^6 \qquad \mathcal{V}' = [\mathrm{Ad}_T]\mathcal{V} \quad \text{or} \quad \mathcal{V}' = \mathrm{Ad}_T(\mathcal{V})$$

$$[\mathcal{V}] \in se(3) \qquad [\mathcal{V}'] = T[\mathcal{V}]T^{-1}$$

Twists

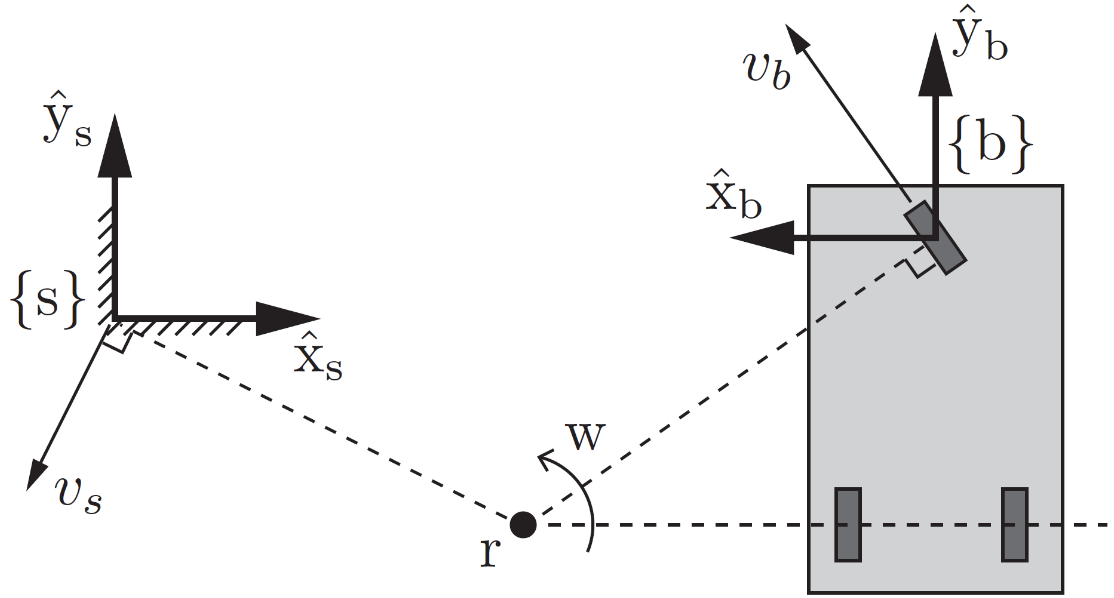
$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s$$

In general

$$\mathcal{V}_c = [\mathrm{Ad}_{T_{cd}}]\mathcal{V}_d, \qquad \mathcal{V}_d = [\mathrm{Ad}_{T_{dc}}]\mathcal{V}_c$$

Twists Example



Top view of a car

- Pure Angular velocity $\mathrm{w} = 2\ \mathrm{rad/s}$

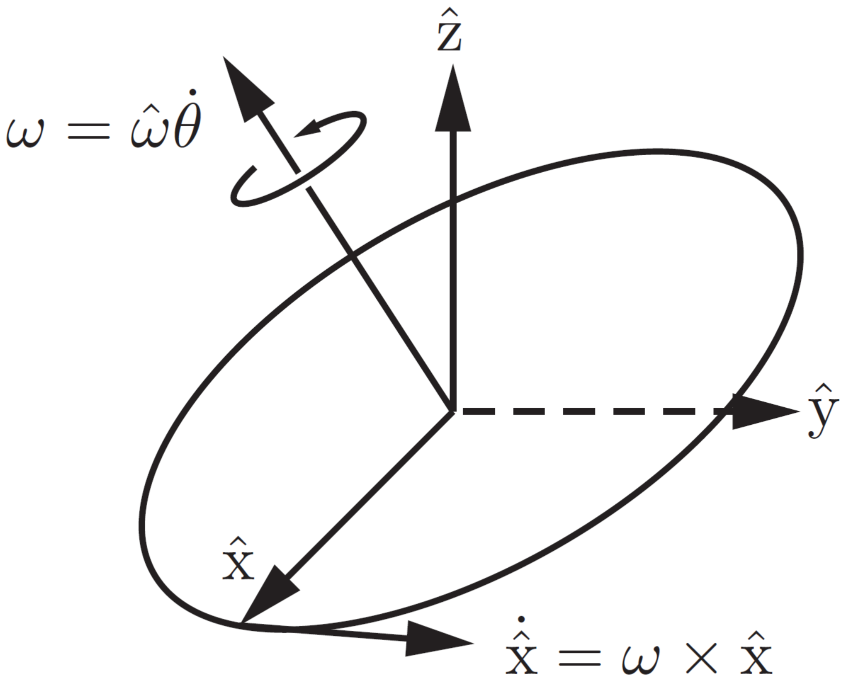
$$r_s = (2, -1, 0) \quad r_b = (2, -1.4, 0)$$

$$\omega_s = (0, 0, 2) \quad \omega_b = (0, 0, -2)$$

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What are the linear velocities? $v_s \quad v_b$

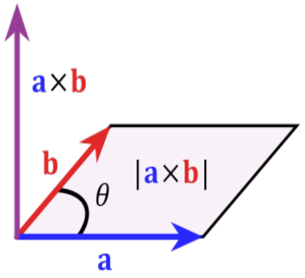
Recall Angular Velocities



- Angular velocity $\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}$

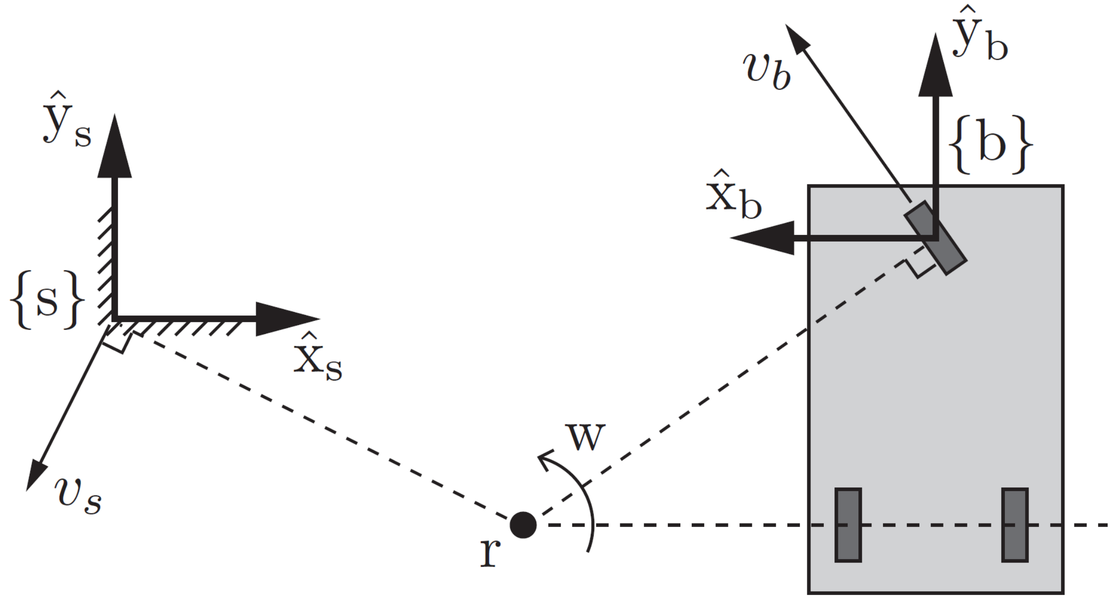
$$\begin{aligned} \dot{\hat{x}} &= \mathrm{w} \times \hat{x}, \\ \dot{\hat{y}} &= \mathrm{w} \times \hat{y}, \\ \dot{\hat{z}} &= \mathrm{w} \times \hat{z}. \end{aligned}$$

Vector cross product



https://en.wikipedia.org/wiki/Cross_product

Twists Example



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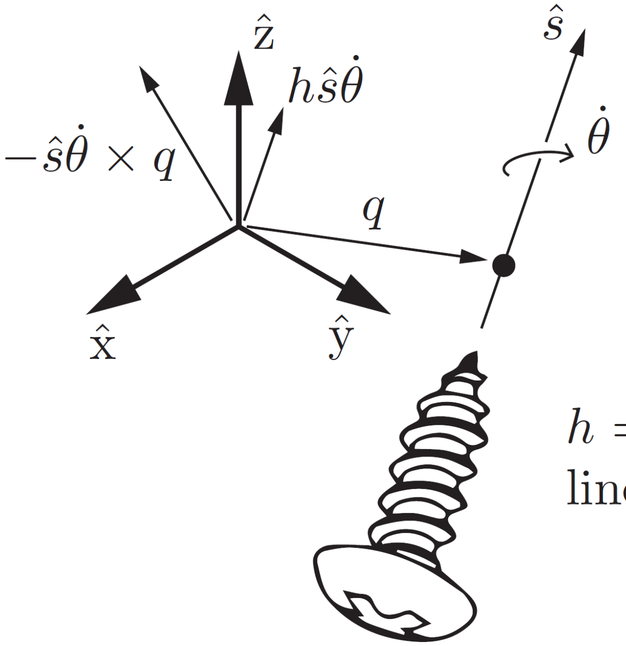
Linear velocity of the car

$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2, -4, 0),$$
$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8, 4, 0),$$

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \qquad \mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix}$$

The Screw Interpretation of a Twist

- Screw axis: motion of a screw
  - Rotating about the axis while translating along the axis



h = pitch =
linear speed/angular speed

Screw axis $\mathcal{S}$ is the collection $\{q, \hat{s}, h\}$

$q \in \mathbb{R}^3$ A point on the axis

Twist about S

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}$$

The Screw Interpretation of a Twist

- For any twist $\mathcal{V} = (\omega, v) \quad \omega \neq 0$
- These exists $\{q, \hat{s}, h\} \quad \dot{\theta}$

$$\hat{s} = \omega / \|\omega\| \qquad \dot{\theta} = \|\omega\| \qquad h = \hat{\omega}^{\mathrm{T}} v / \dot{\theta}$$

portion of v parallel to the screw axis

$-\hat{s}\dot{\theta} \times q$ provides the portion of v orthogonal to the screw axis

If $\omega = 0$

$$\hat{s} = v / \|v\|$$

$\dot{\theta}$ is interpreted as the linear velocity $\|v\|$ along $\hat{s}$

The Screw Interpretation of a Twist

- Another representation of the screw axis

If $\omega \neq 0$ $\qquad \mathcal{S} = \mathcal{V}/\|\omega\| = (\omega/\|\omega\|, v/\|\omega\|)$

$\mathcal{V} = (\omega, v)$ $\qquad \dot{\theta} = \|\omega\| \qquad \mathcal{S}\dot{\theta} = \mathcal{V}$

If $\omega = 0$ $\qquad \mathcal{S} = \mathcal{V}/\|v\| = (0, v/\|v\|)$

$\dot{\theta} = \|v\| \qquad \mathcal{S}\dot{\theta} = \mathcal{V}$

Screw Axis

- A screw axis is a normalized twist $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$ $\quad \mathcal{S}\dot{\theta} = \mathcal{V}$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3) \qquad [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$

$$\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \qquad \mathcal{S}_b = [\mathrm{Ad}_{T_{ba}}]\mathcal{S}_a$$

Summary

- Twists
- Screw Axis

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017
- M. Ceccarelli. Screw axis defined by Giulio Mozzi in 1763 and early studies on helicoidal motion. Mechanism and Machine Theory, 35:761-770, 2000.
- J. M. McCarthy. Introduction to Theoretical Kinematics. MIT Press, Cambridge, MA, 1990.