Geometric Primitives and Transformations

CS 4391 Introduction Computer Vision

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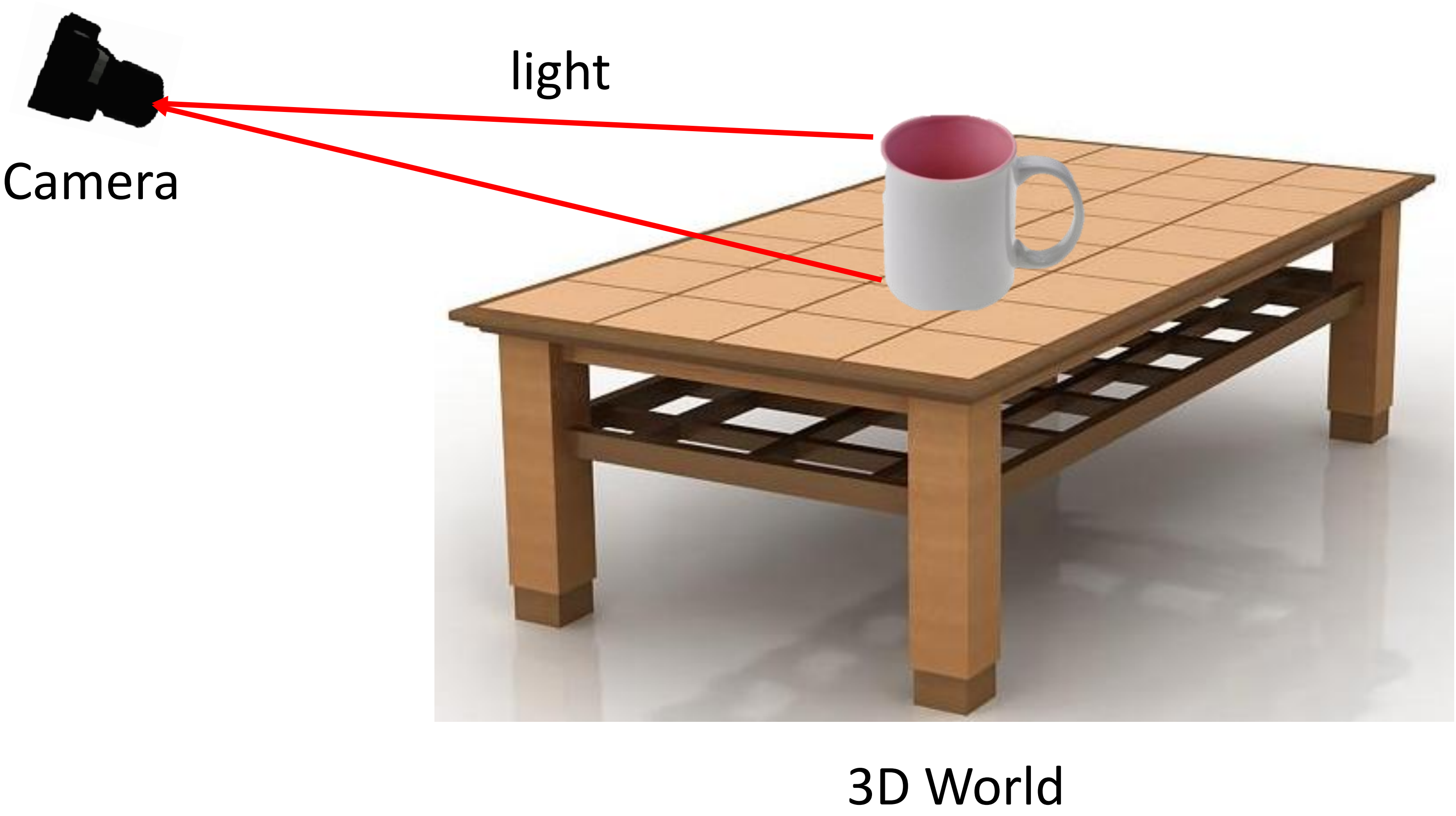
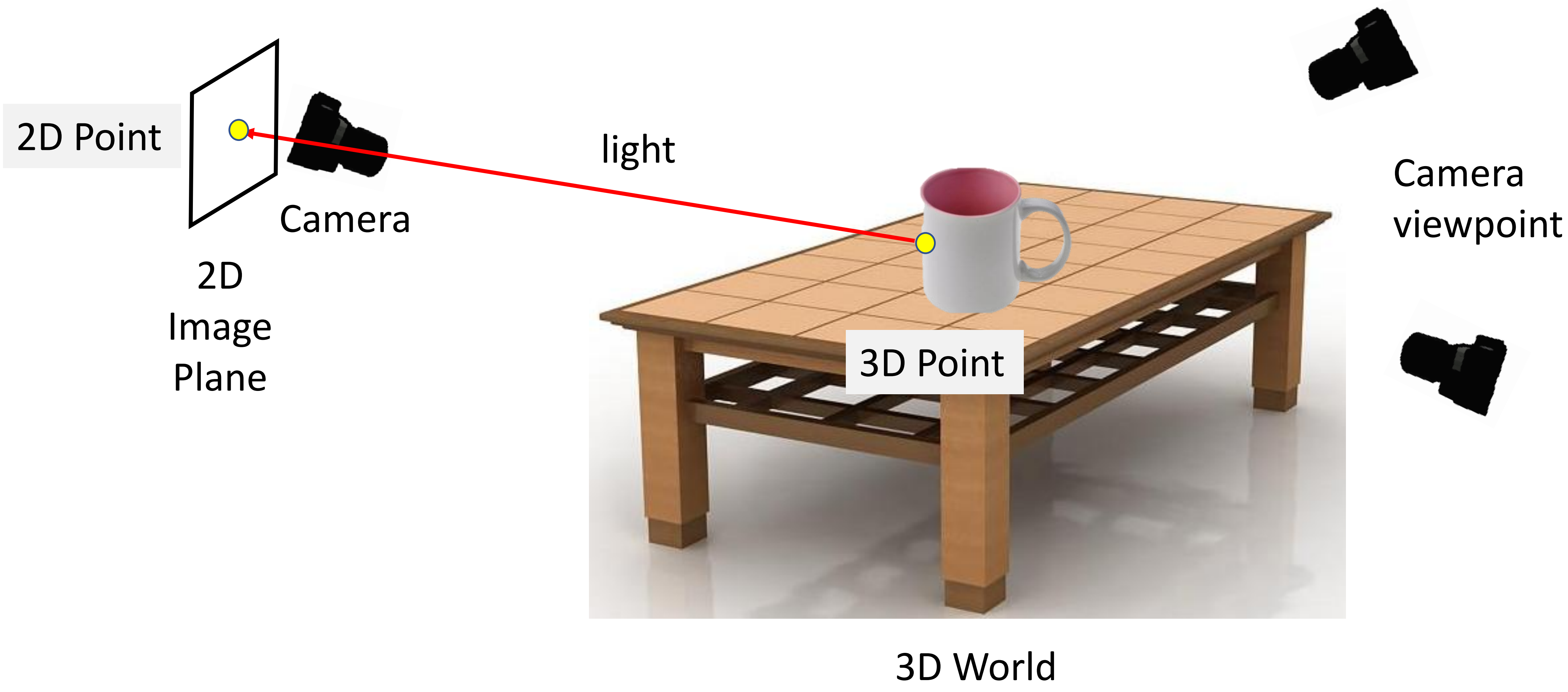
The University of Texas at Dallas

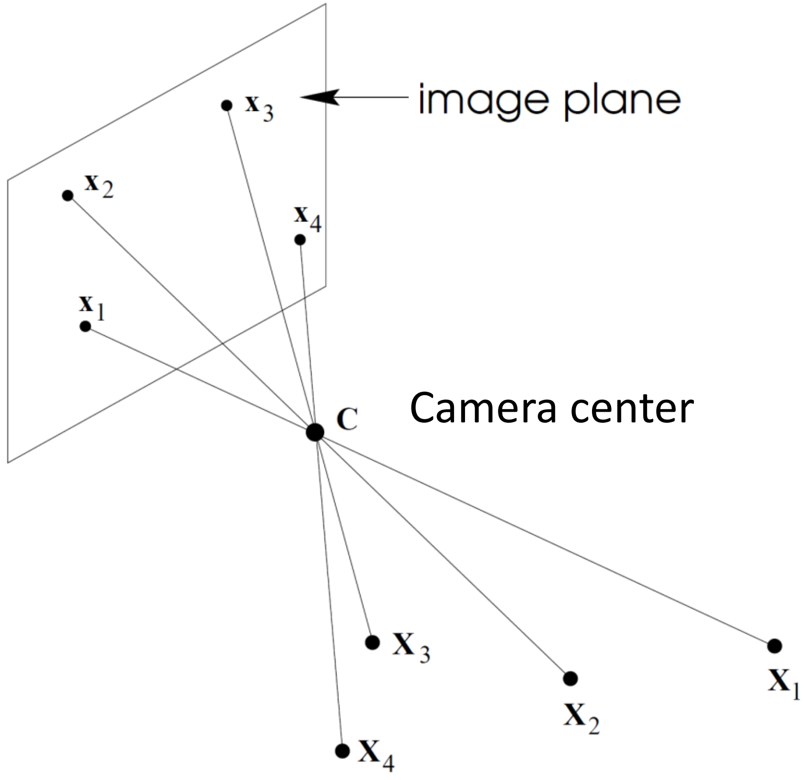
How are Images Generated?



Geometry in Image Generation



2D Points and 3D Points



- A 2D point is usually used to indicate pixel coordinates of a pixel

$$\mathbf{x} = (x, y) \in \mathcal{R}^2 \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- A 3D point in the real world

$$\mathbf{X} = (x, y, z) \in \mathcal{R}^3 \qquad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Homogeneous Coordinates

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = w \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene coordinates

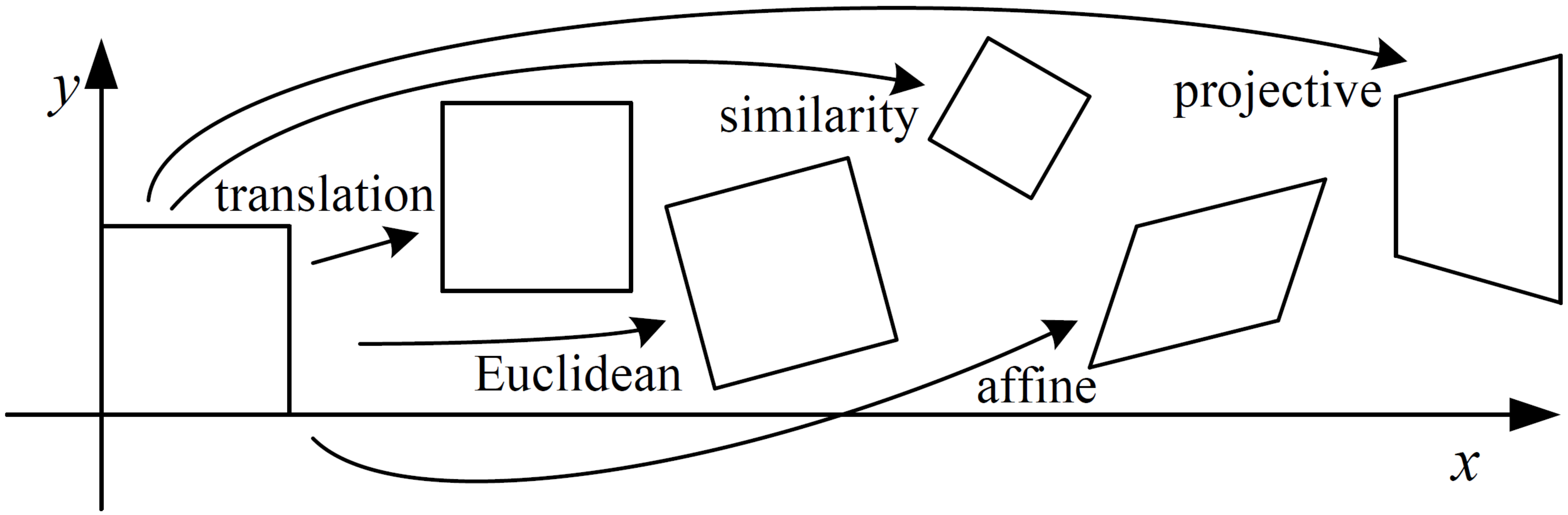
Up to scale

Conversion

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

2D Transformations



2D Translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

Homogeneous coordinate

$$\mathbf{x}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

$$2 \times 3$$

$$\bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \bar{\mathbf{x}}$$

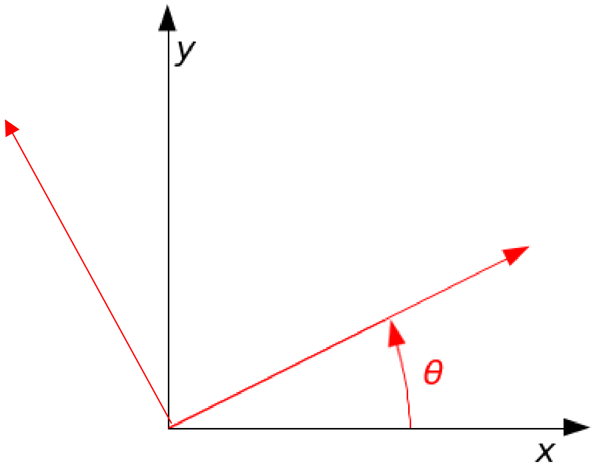
$$3 \times 3$$

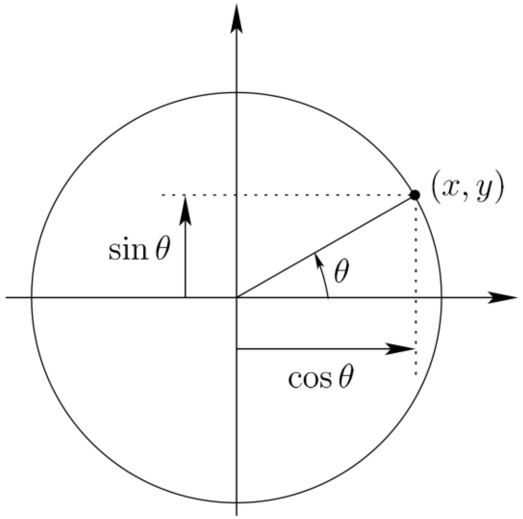
augmented vector $\bar{\mathbf{x}} = (x, y, 1)$

2D Rotation

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$





$$[\hat{\mathrm{x}}_{\mathrm{b}} \quad \hat{\mathrm{y}}_{\mathrm{b}}] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

orthonormal rotation matrix

$$\mathbf{R}\mathbf{R}^T = \mathbf{I} \text{ and } |\mathbf{R}| = 1$$

2D Euclidean Transformation

- 2D Rotation + 2D translation

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

2D Euclidean Transformation

- 2D Rotation + 2D translation

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{x}' = \underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}}_{2 \times 3} \bar{\mathbf{x}}$$

$$\bar{\mathbf{x}} = (x, y, 1)$$

- Degree of freedom (DOF)
    - The maximum number of logically independent values
    - 2D Rotation?
    - 2D Euclidean transformation?

2D Similarity Transformation

- Scaled 2D rotation + 2D translation

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}} = \begin{bmatrix} a & -b & t_x \\ b & a & t_y \end{bmatrix} \bar{\mathbf{x}} \qquad \bar{\mathbf{x}} = (x, y, 1)$$

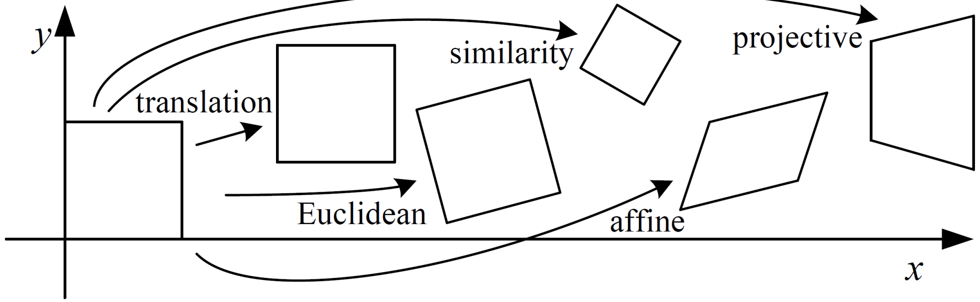
The similarity transform preserves angles between lines.

2D Affine Transformation

- Arbitrary 2x3 matrix

$$\mathbf{x}' = \mathbf{A}\bar{\mathbf{x}} \qquad \bar{\mathbf{x}} = (x, y, 1)$$

$$\mathbf{x}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \bar{\mathbf{x}}$$



Parallel lines remain parallel under affine transformations.

2D Projective Transformation

- Also called perspective transform or homography

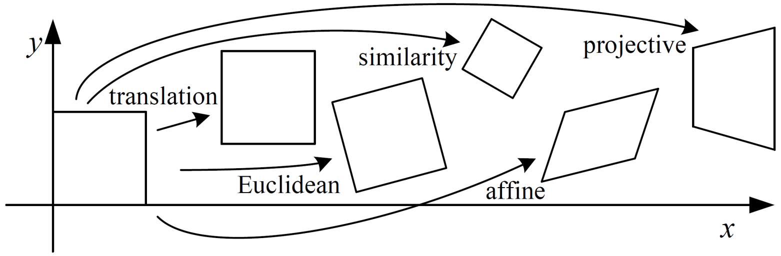
$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}}$$

homogeneous coordinates

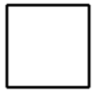
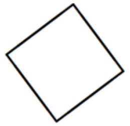
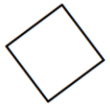
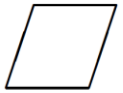
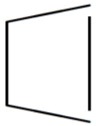
3×3 $\tilde{\mathbf{H}}$ is only defined up to a scale

$$x' = \frac{h_{00}x + h_{01}y + h_{02}}{h_{20}x + h_{21}y + h_{22}} \quad \text{and} \quad y' = \frac{h_{10}x + h_{11}y + h_{12}}{h_{20}x + h_{21}y + h_{22}}$$

Perspective transformations preserve straight lines



Hierarchy of 2D Transformations

Transformation	**Matrix**	**# DoF**	**Preserves**	**Icon**
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

3D Translation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

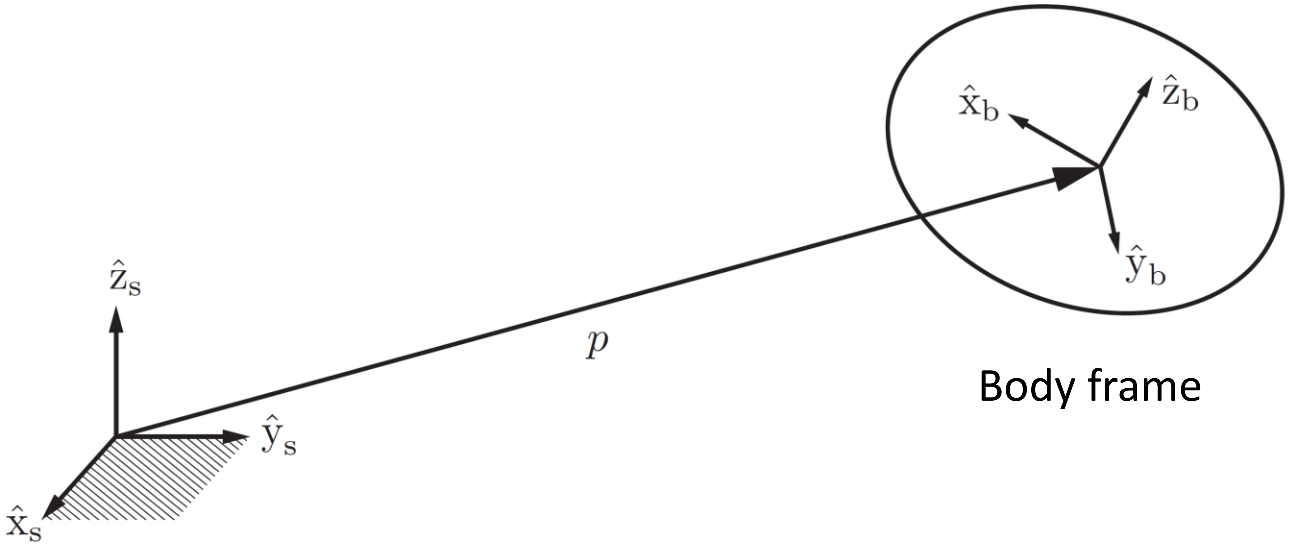
$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

$$3 \times 4$$

augmented vector $\bar{\mathbf{x}} = (x, y, z, 1)$

3D Rotation



Body frame

Fixed frame

- Axes of the body frame

$$
\begin{aligned}
\hat{x}_b &= r_{11}\hat{x}_s + r_{21}\hat{y}_s + r_{31}\hat{z}_s, \\
\hat{y}_b &= r_{12}\hat{x}_s + r_{22}\hat{y}_s + r_{32}\hat{z}_s, \\
\hat{z}_b &= r_{13}\hat{x}_s + r_{23}\hat{y}_s + r_{33}\hat{z}_s.
\end{aligned}
$$

$$
R = [\hat{x}_b \;\; \hat{y}_b \;\; \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

Write as column vectors

Rotation matrix

We will focus on 3D rotations in next lectures.

3D Euclidean Transformation SE(3)

- 3D Rotation + 3D translation

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

$$3 \times 4$$

$$\bar{\mathbf{x}} = (x, y, z, 1)$$

orthonormal rotation matrix

$$\mathbf{R}\mathbf{R}^T = \mathbf{I} \text{ and } |\mathbf{R}| = 1$$

$$3 \times 3$$

3D Similarity Transformation

- Scaled 3D rotation + 3D translation

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \underset{3 \times 4}{\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}} \bar{\mathbf{x}} \qquad \bar{\mathbf{x}} = (x, y, z, 1)$$

This transformation preserves angles between lines and planes.

3D Affine Transformation

$$\mathbf{x}' = \mathbf{A}\bar{\mathbf{x}} \qquad \bar{\mathbf{x}} = (x, y, z, 1)$$

$$\mathbf{x}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \end{bmatrix} \bar{\mathbf{x}}$$

$$3 \times 4$$

Parallel lines and planes remain parallel under affine transformations.

3D Projective Transformation

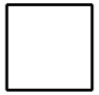
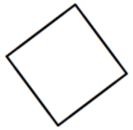
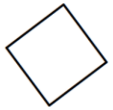
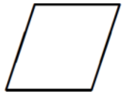
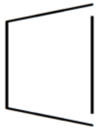
- Also called 3D perspective transform or homography

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}}$$ homogeneous coordinates

4×4 $\tilde{\mathbf{H}}$ is only defined up to a scale

- Perspective transformations preserve straight lines

3D Transformations

Transformation	**Matrix**	**# DoF**	**Preserves**	**Icon**
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{3\times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$	7	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{3\times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{4\times 4}$	15	straight lines	

Further Reading

- Section 2.1, Computer Vision, Richard Szeliski

- Chapter 2 and 3, Multiple View Geometry in Computer Vision, Richard Hartley and Andrew Zisserman