# Structure from Motion I

CS 4391 Introduction Computer Vision

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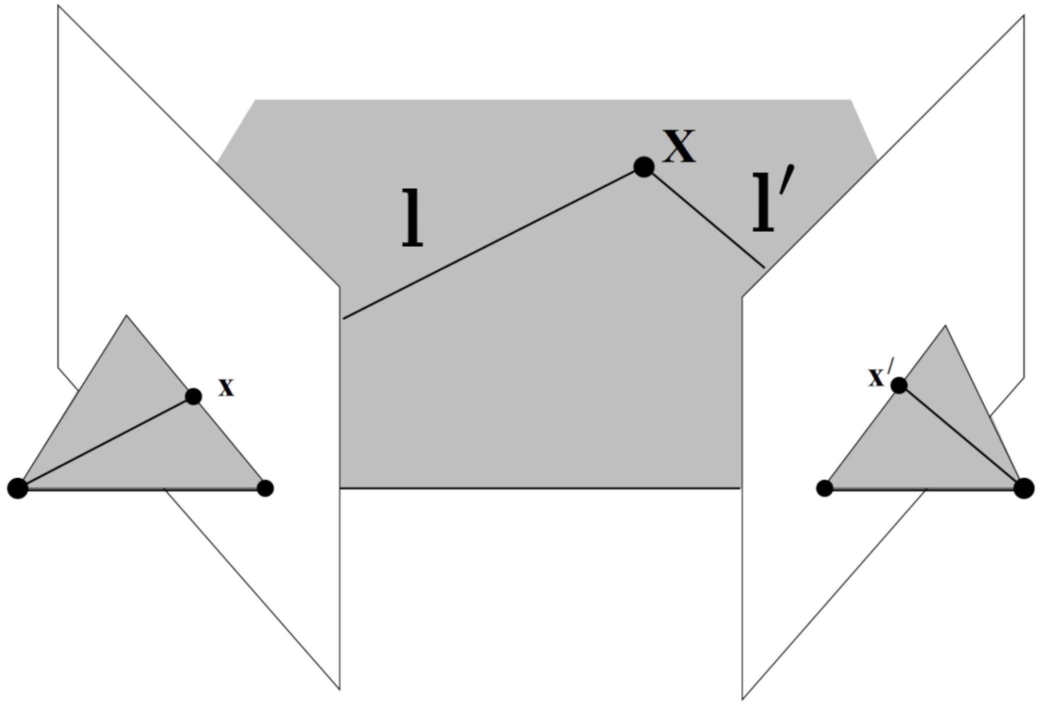
# Triangulation

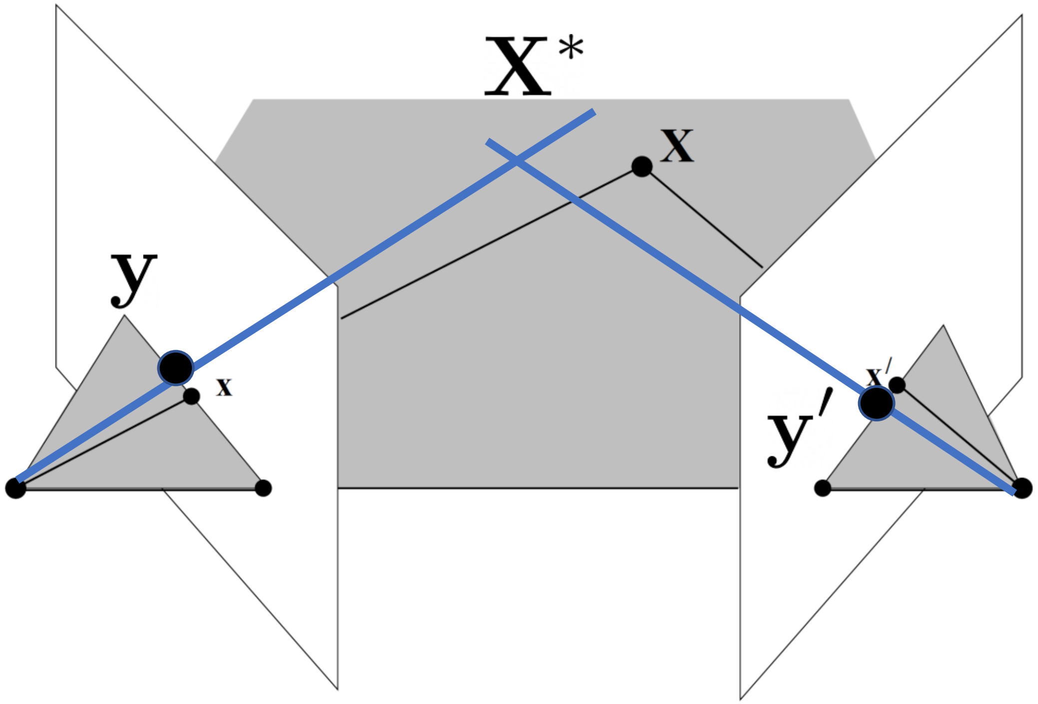
- Compute the 3D point given image correspondences




Intersection of two backprojected lines

$$\mathbf{X} = \mathbf{l} \times \mathbf{l}'$$

# Triangulation




- In practice, we find the correspondences
  $\mathbf{y} \;\; \mathbf{y}'$
- The backprojected lines may not intersect
- Find $X^*$ that minimizes

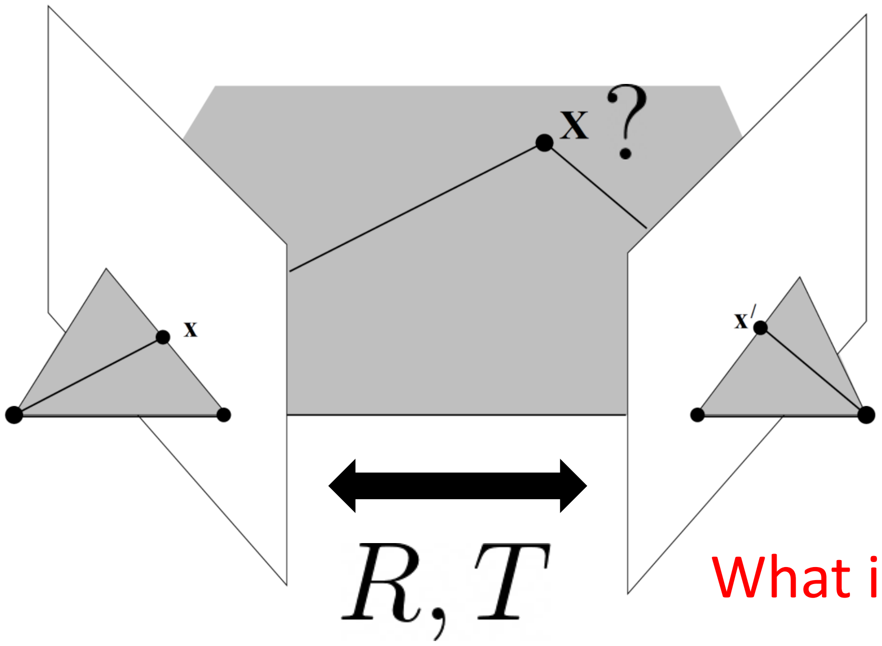
$$d(\mathbf{y}, P\mathbf{X}^*) + d(\mathbf{y}', P'\mathbf{X}^*)$$

Projection matrix

# Triangulation

- Idea: using images from different views and feature matching

- Triangulation from pixel correspondences to compute 3D location




Given $\mathbf{x} \longleftrightarrow \mathbf{x}'$

Intersection of two backprojected lines

$$\mathbf{X} = \mathbf{l} \times \mathbf{l}'$$

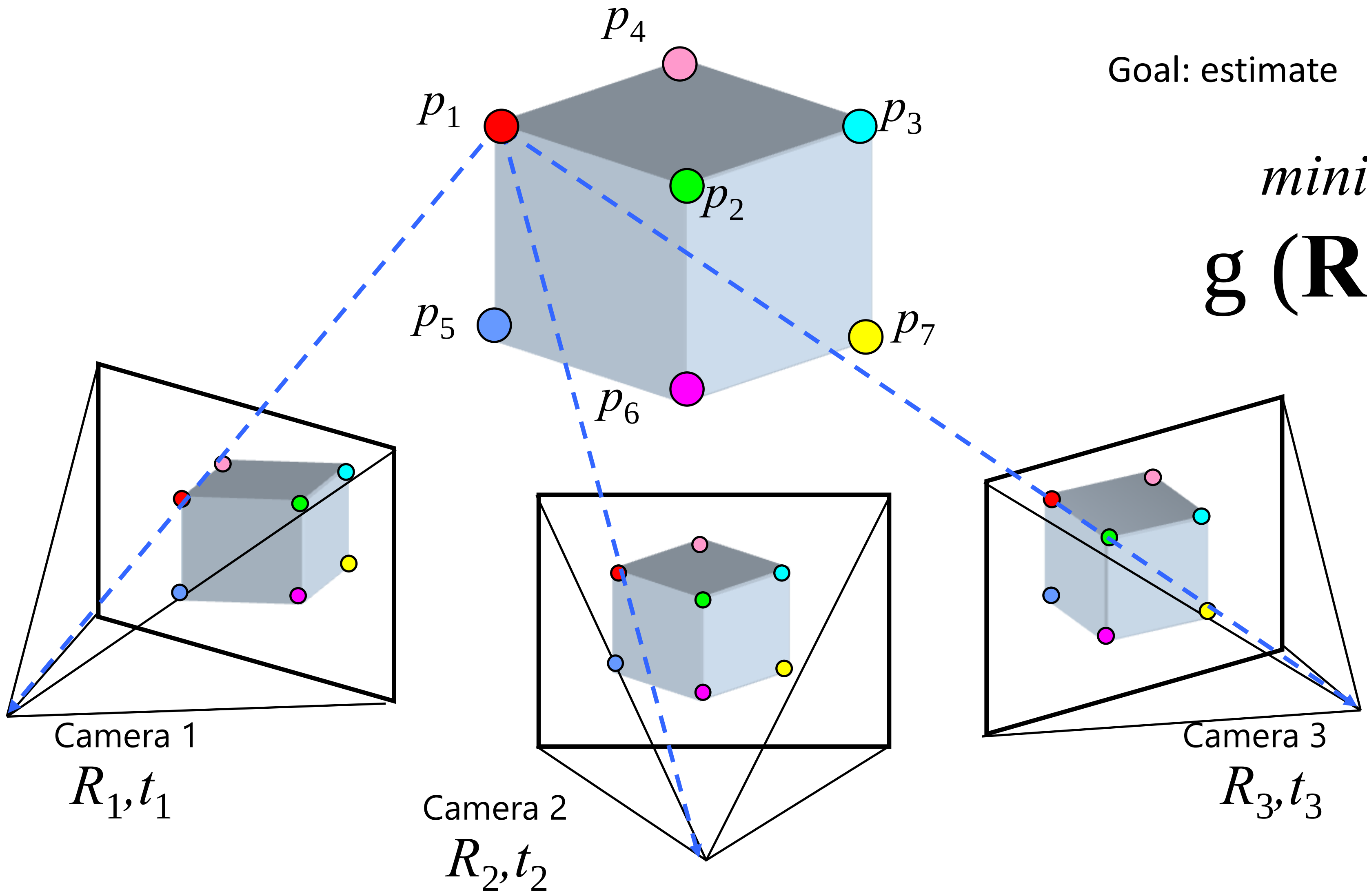
What if unknow camera pose?

# Structure from Motion

- Input
  - A set of images from different views

- Output
  - 3D Locations of all feature points in a world frame
  - Camera poses of the images

# Structure from motion




Goal: estimate $\mathbf{R}, \mathbf{T}, \mathbf{P}$

*minimize*

$$\mathbf{g}\left(\mathbf{R}, \mathbf{T}, \mathbf{P}\right)$$

# Structure from Motion

- Minimize sum of squared reprojection errors

$$g(\mathbf{X}, \mathbf{R}, \mathbf{T}) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \cdot \left\| \underbrace{\mathbf{P}(\mathbf{x}_i, \mathbf{R}_j, \mathbf{t}_j)}_{\text{predicted image location}} - \underbrace{\begin{bmatrix} u_{i,j} \\ v_{i,j} \end{bmatrix}}_{\text{observed image location}} \right\|^2$$

m points, n images

*predicted* image location

*observed* image location

*Indicator variable:*
is point i visible in image j?

Projection

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathbf{Rx} + \mathbf{t}$$

$$u' = f_x \frac{x'}{z'} + p_x$$

$$v' = f_y \frac{y'}{z'} + p_y$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \mathbf{P}(\mathbf{x}, \mathbf{R}, \mathbf{t})$$

# Structure from Motion

- How to minimize

$$g(\mathbf{X},\mathbf{R},\mathbf{T}) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij} \cdot \left\| \mathbf{P}(\mathbf{x}_i, \mathbf{R}_j, \mathbf{t}_j) - \begin{bmatrix} u_{i,j} \\ v_{i,j} \end{bmatrix} \right\|^2$$

- A non-linear least squares problem (why?)
  - E.g. Levenberg-Marquardt

# The Levenberg-Marquardt Algorithm

- Nonlinear least squares $\hat{\boldsymbol{\beta}} \in \operatorname{argmin}_{\boldsymbol{\beta}} S(\boldsymbol{\beta}) \equiv \operatorname{argmin}_{\boldsymbol{\beta}} \sum_{i=1}^{m} [y_i - f(x_i, \boldsymbol{\beta})]^2$

  $n \times 1$

- An iterative algorithm
  - Start with an initial guess $\beta_0$
  - For each iteration $\beta \leftarrow \beta + \delta$

- How to get $\delta$?
  - Linear approximation $f(x_i, \boldsymbol{\beta} + \boldsymbol{\delta}) \approx f(x_i, \boldsymbol{\beta}) + \mathbf{J}_i \boldsymbol{\delta}$   $\mathbf{J}_i = \frac{\partial f(x_i, \boldsymbol{\beta})}{\partial \boldsymbol{\beta}}$   $1 \times n$
  - Find $\delta$ to minimize the objective $S(\boldsymbol{\beta} + \boldsymbol{\delta}) \approx \sum_{i=1}^{m} [y_i - f(x_i, \boldsymbol{\beta}) - \mathbf{J}_i \boldsymbol{\delta}]^2$

Best to minimize the objective

Wikipedia

# The Levenberg-Marquardt Algorithm

- Vector notation for $S(\boldsymbol{\beta}+\boldsymbol{\delta}) \approx \sum_{i=1}^{m}\left[y_i - f(x_i, \boldsymbol{\beta}) - \mathbf{J}_i \boldsymbol{\delta}\right]^2$

$$
\begin{aligned}
S(\boldsymbol{\beta}+\boldsymbol{\delta}) &\approx \|\mathbf{y} - \mathbf{f}(\boldsymbol{\beta}) - \mathbf{J}\boldsymbol{\delta}\|^2 \\
&= \left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta}) - \mathbf{J}\boldsymbol{\delta}\right]^{\mathrm{T}} \left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta}) - \mathbf{J}\boldsymbol{\delta}\right] \\
&= \left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})\right]^{\mathrm{T}} \left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})\right] - \left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})\right]^{\mathrm{T}} \mathbf{J}\boldsymbol{\delta} - (\mathbf{J}\boldsymbol{\delta})^{\mathrm{T}} \left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})\right] + \boldsymbol{\delta}^{\mathrm{T}}\mathbf{J}^{\mathrm{T}}\mathbf{J}\boldsymbol{\delta} \\
&= \left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})\right]^{\mathrm{T}} \left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})\right] - 2\left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})\right]^{\mathrm{T}} \mathbf{J}\boldsymbol{\delta} + \boldsymbol{\delta}^{\mathrm{T}}\mathbf{J}^{\mathrm{T}}\mathbf{J}\boldsymbol{\delta}.
\end{aligned}
$$

Take derivation with respect to $\delta$ and set to zero $\left(\mathbf{J}^{\mathrm{T}}\mathbf{J}\right)\boldsymbol{\delta} = \mathbf{J}^{\mathrm{T}}\left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})\right]$ https://www.cs.ubc.ca/~schmidtm/Courses/340-F16/linearQuadraticGradients.pdf

Levenberg's contribution $\left(\mathbf{J}^{\mathrm{T}}\mathbf{J} + \lambda\mathbf{I}\right)\boldsymbol{\delta} = \mathbf{J}^{\mathrm{T}}\left[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})\right]$ damped version

$$\beta \leftarrow \beta + \delta$$

Wikipedia

# Structure from Motion

$$g(\mathbf{X}, \mathbf{R}, \mathbf{T}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{w_{ij}}_{\substack{\textit{indicator variable}: \\ \text{is point } i \text{ visible in image } j\text{ ?}}} \cdot \left\| \underbrace{\mathbf{P}(\mathbf{x}_i, \mathbf{R}_j, \mathbf{t}_j)}_{\substack{\textit{predicted} \\ \text{image location}}} - \underbrace{\begin{bmatrix} u_{i,j} \\ v_{i,j} \end{bmatrix}}_{\substack{\textit{observed} \\ \text{image location}}} \right\|^2$$

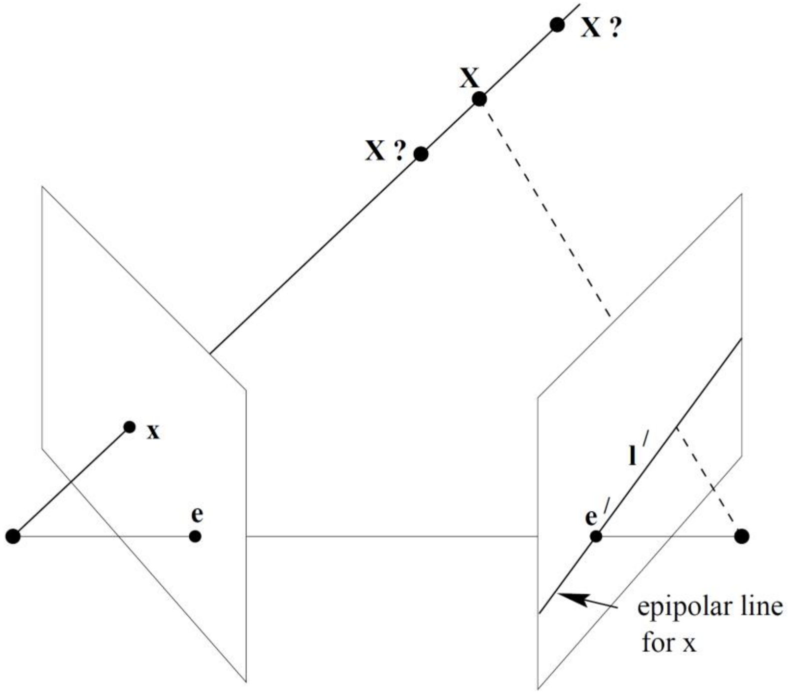
$$\beta = (\mathbf{X}, \mathbf{R}, \mathbf{T})$$

How to get the initial estimation $\beta_0$ ?

Random guess is not a good idea.

# Matching Two Views

- Fundamental matrix




$\mathbf{x}'$ is on the epiploar line $\mathbf{l}' = F\mathbf{x}$

$$\mathbf{x}'^T F \mathbf{x} = 0$$

The 8-point algorithm

# Further Reading

- Chapter 11, Computer Vision, Richard Szeliski
- Build Rome in One Day https://grail.cs.washington.edu/rome/
- Structure from Motion Revisited https://colmap.github.io/index.html