Epipolar Geometry

CS 4391 Introduction Computer Vision

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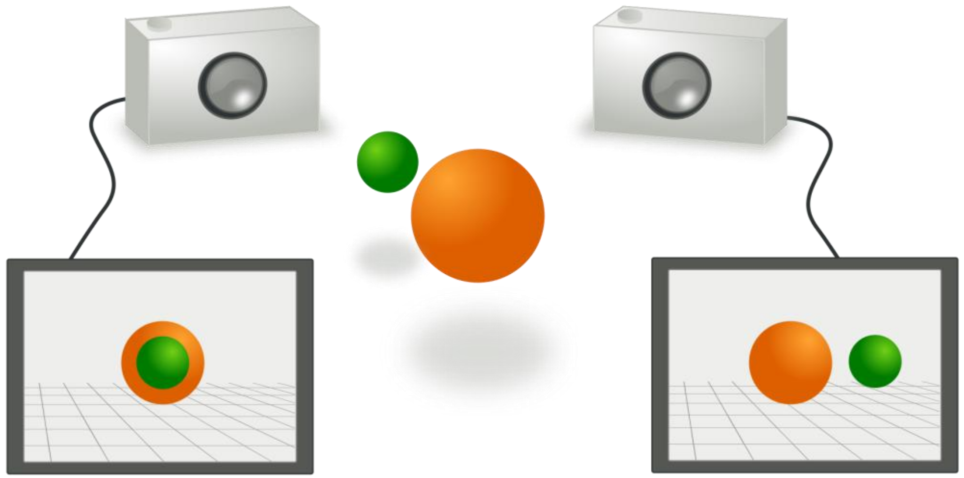
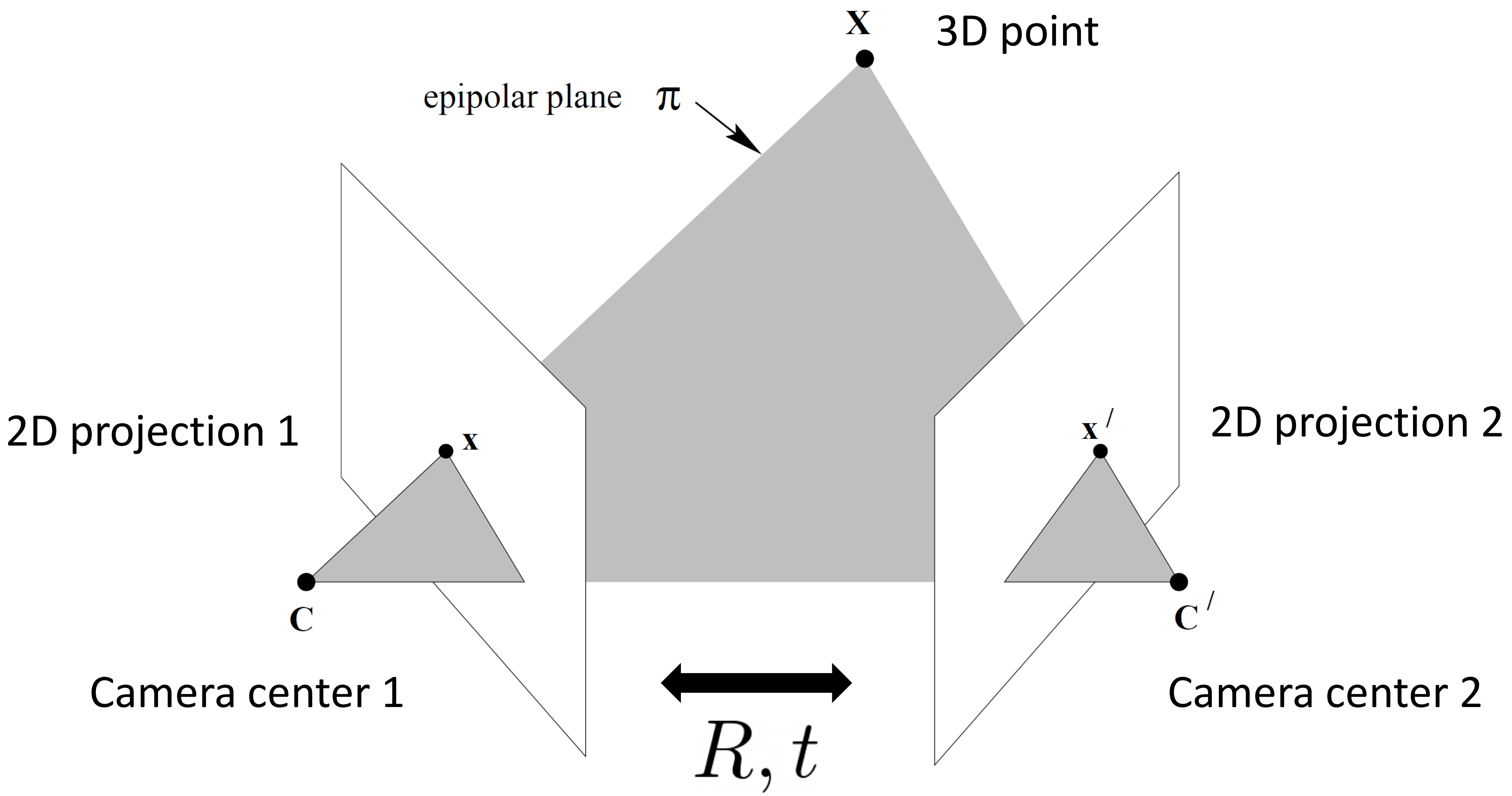
Epipolar Geometry

- The geometry of stereo vision
  - Given 2D images of two views
  - What is the relationship between pixels of the images?
  - Can we recover the 3D structure of the world from the 2D images?

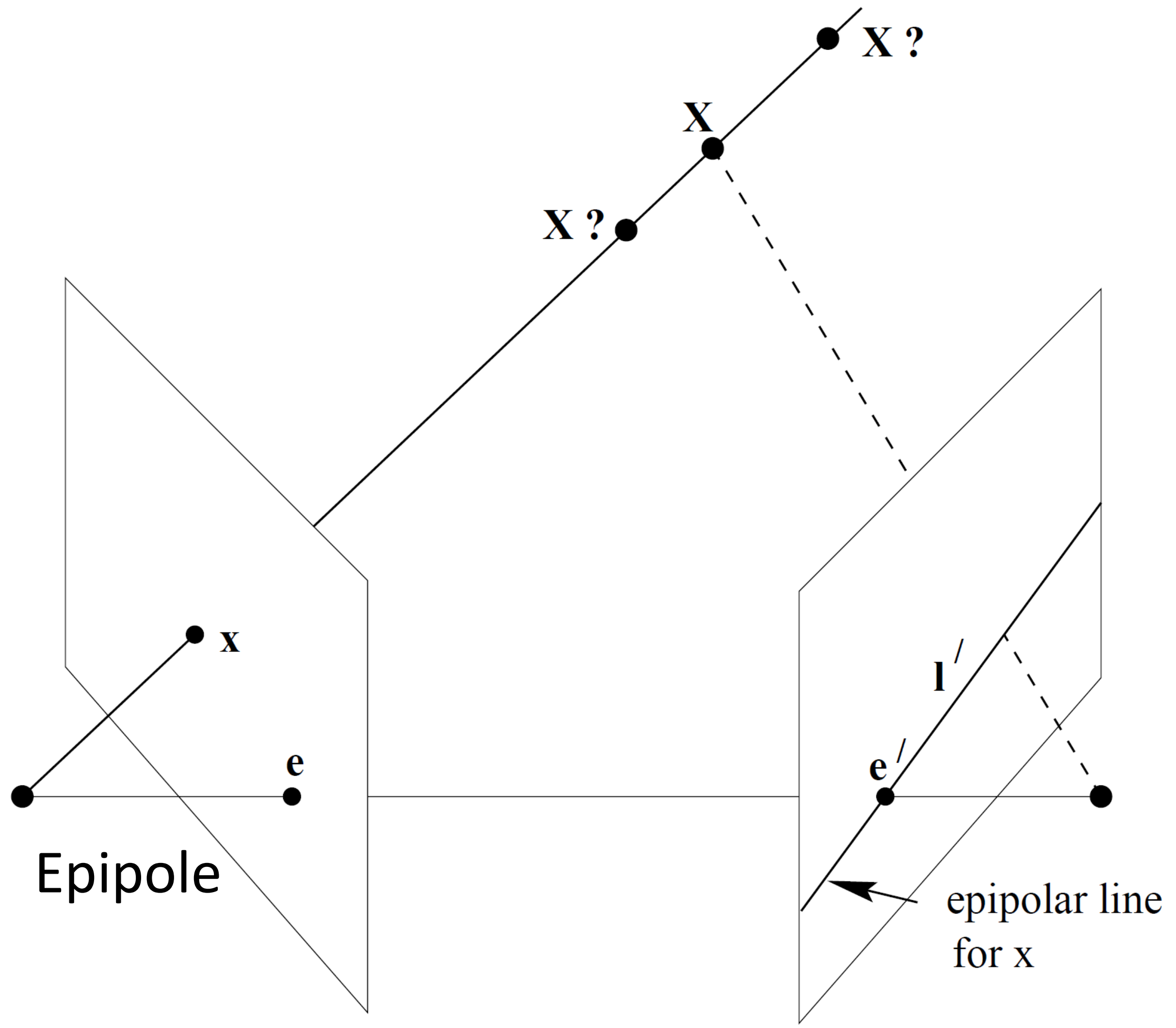


Wikipedia

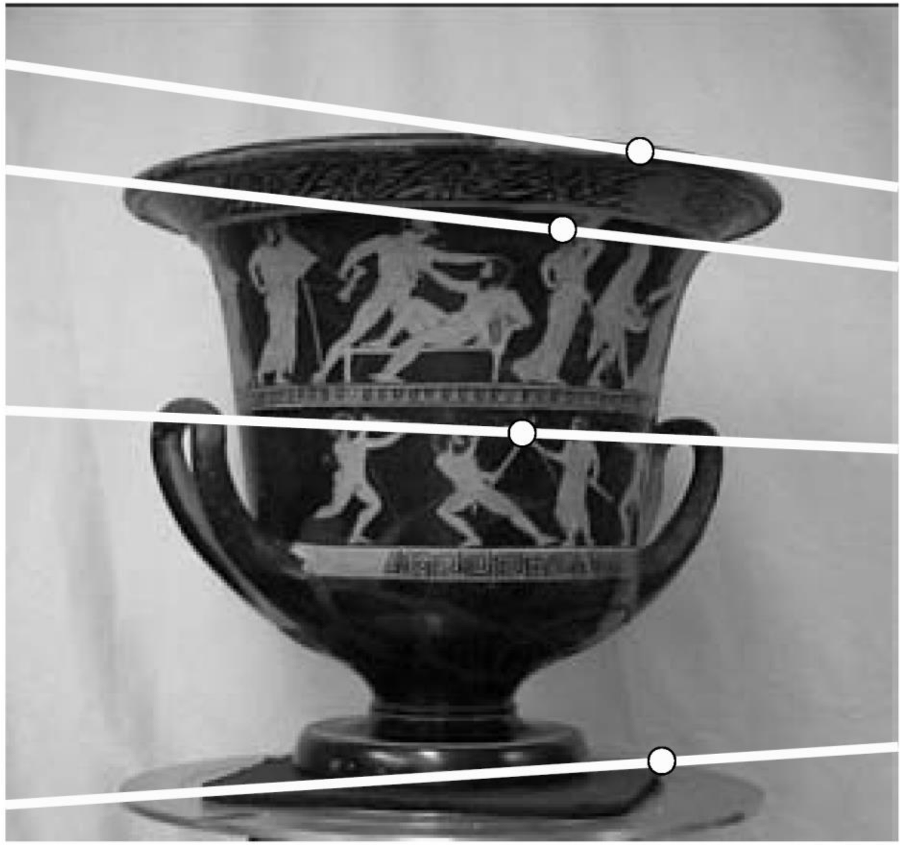
Epipolar Geometry



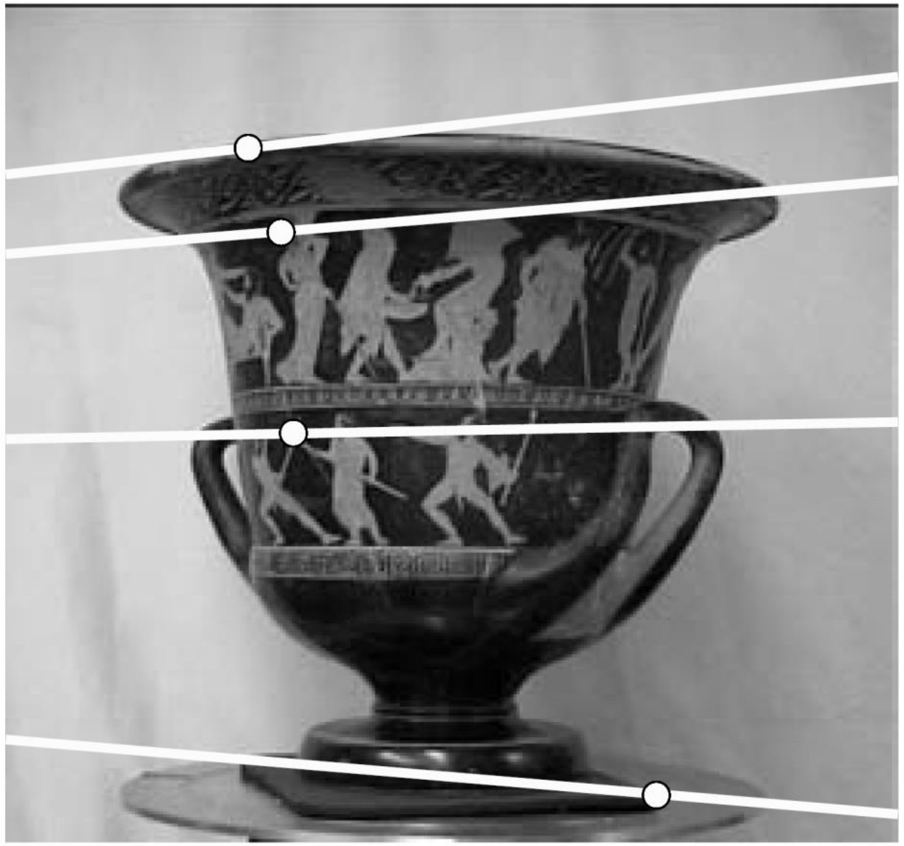
Epipolar Geometry



Epipolar Geometry



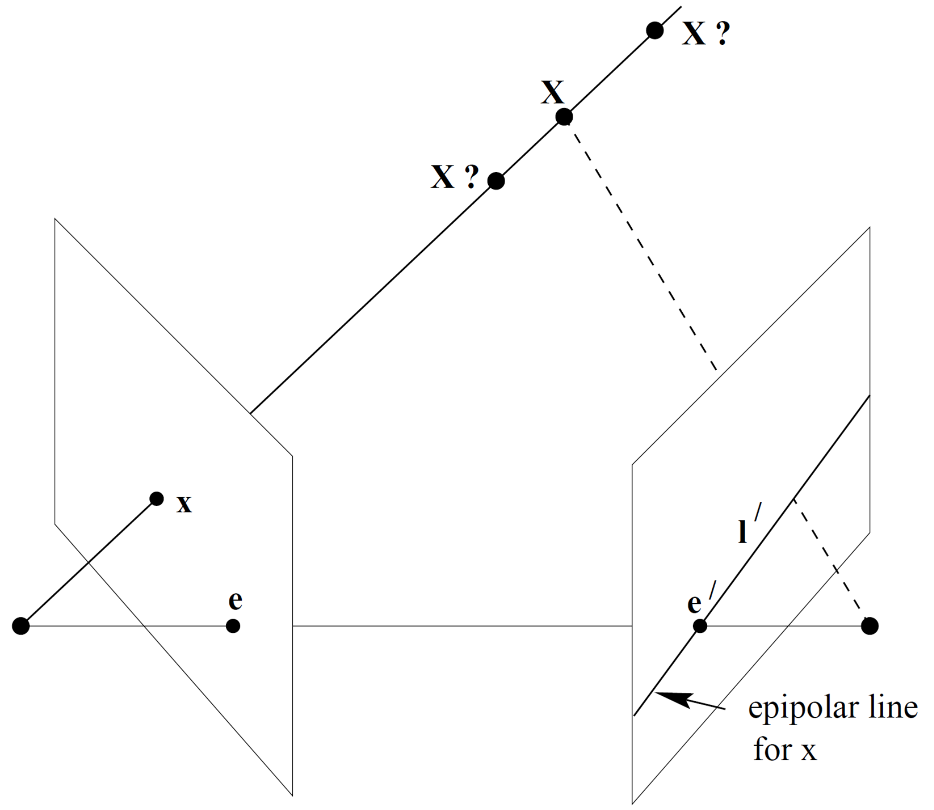
Epipolar lines



Rotation and Translation
between two views

Epipolar Geometry

- What is the mapping for a point in one image to its epipolar line?



$$\mathbf{x} \mapsto \mathbf{l}'$$

2D Lines

- A line in a 2D plane $ax + by + c = 0$ $\quad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- It is parameterized by $\mathbf{l} = (a, b, c)^T$ Homogeneous Coordinates

  $k(a, b, c)^T$ represents the same line for nonzero k
- Line equation

$$\mathbf{x}^T\mathbf{l} = 0 \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

A Line Joining two Points

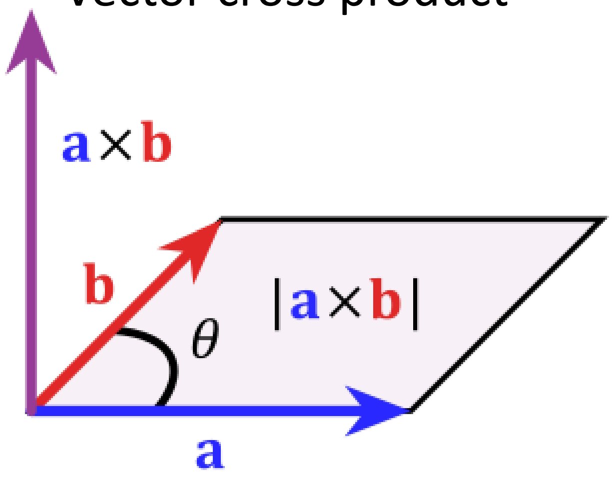
$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$\mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

$$\mathbf{x} \cdot (\mathbf{x} \times \mathbf{x}') = \mathbf{x}' \cdot (\mathbf{x} \times \mathbf{x}') = 0$$

$$\mathbf{x}^T \mathbf{l} = \mathbf{x}'^T \mathbf{l} = 0$$

Vector cross product



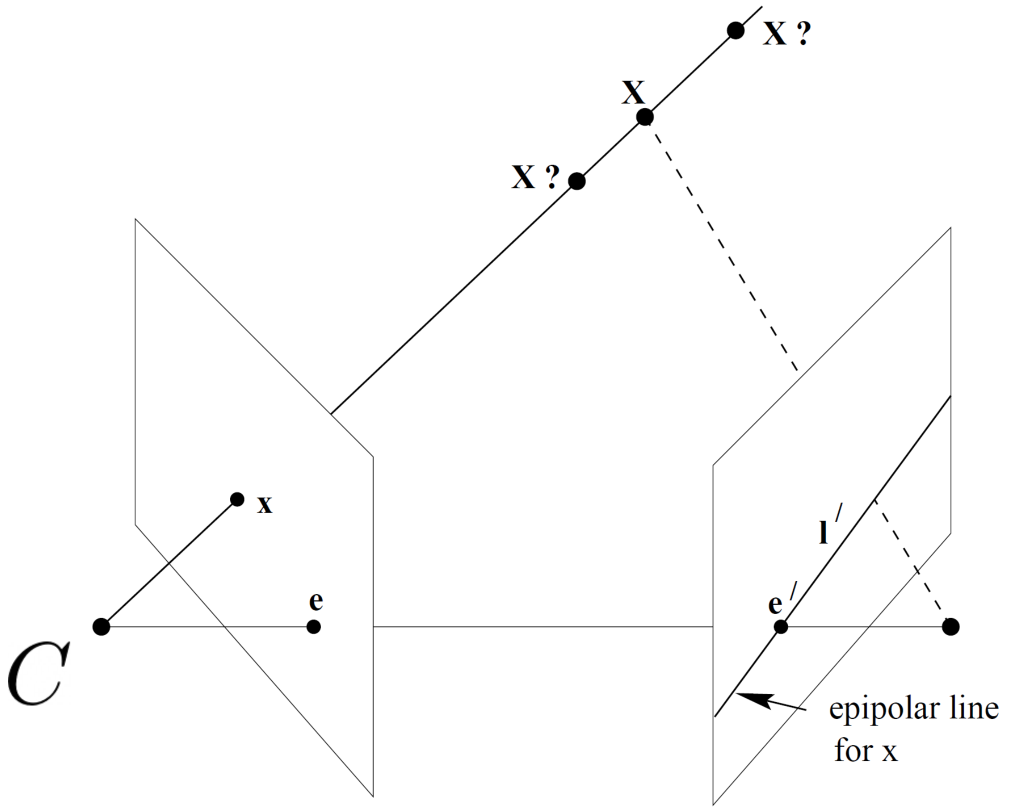
$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \, \mathbf{n}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Vector dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

Fundamental Matrix



- Recall camera projection

$$P = K[R|\mathbf{t}]$$

$$\mathbf{x} = P\mathbf{X}$$ Homogeneous coordinates

- Backprojection

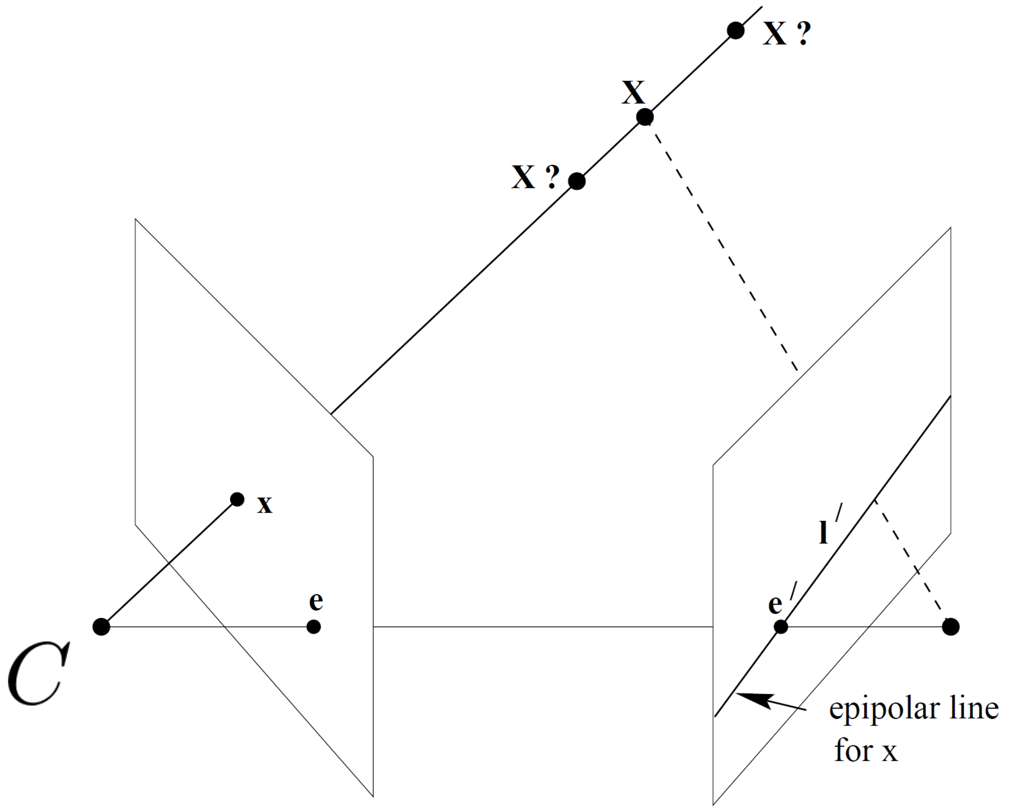
$P^+\mathbf{x}$ and C are two points on the ray

P^+ is the pseudo-inverse of P, $PP^+ = I$

$$P^+ = P^T(PP^T)^{-1}$$

$$\mathbf{X}(\lambda) = (1-\lambda)P^+\mathbf{x} + \lambda C$$

Fundamental Matrix



$P^+\mathbf{x}$ and C are two points on the ray

- Project to the other image

$$P'P^+\mathbf{x} \quad \text{and} \quad P'C$$

- Epipolar line

$$\mathbf{l}' = (P'C) \times (P'P^+\mathbf{x})$$

Epipole $\mathbf{e}' = (P'C)$

Skew-symmetric Matrix

$$x = [x_1 \;\; x_2 \;\; x_3]^{\mathrm{T}} \in \mathbb{R}^3$$

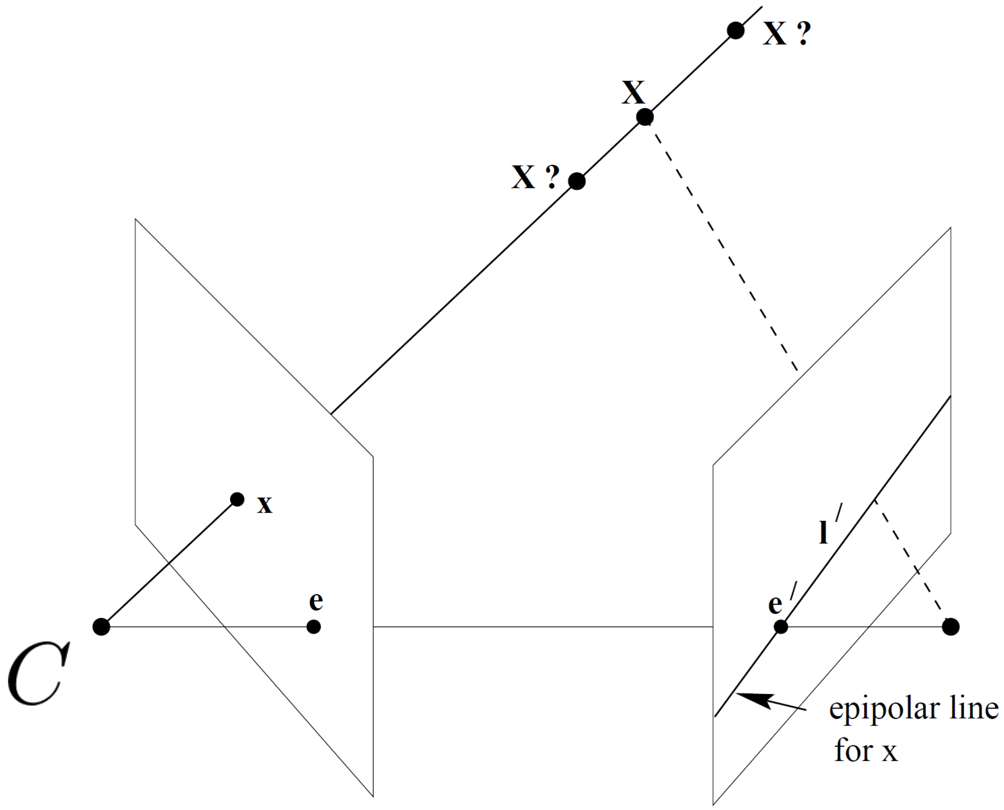
$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$$\mathbf{x} \times \mathbf{y} = [\mathbf{x}]\mathbf{y}$$

$$[x] = -[x]^{\mathrm{T}}$$

https://en.wikipedia.org/wiki/Skew-symmetric_matrix

Fundamental Matrix



- Epipolar line

$$\mathbf{l}' = (P'C) \times (P'P^{+}\mathbf{x})$$

Epipole $\mathbf{e}' = (P'C)$

$$\mathbf{l}' = [\mathbf{e}']_{\times}(P'P^{+}\mathbf{x}) = F\mathbf{x}$$

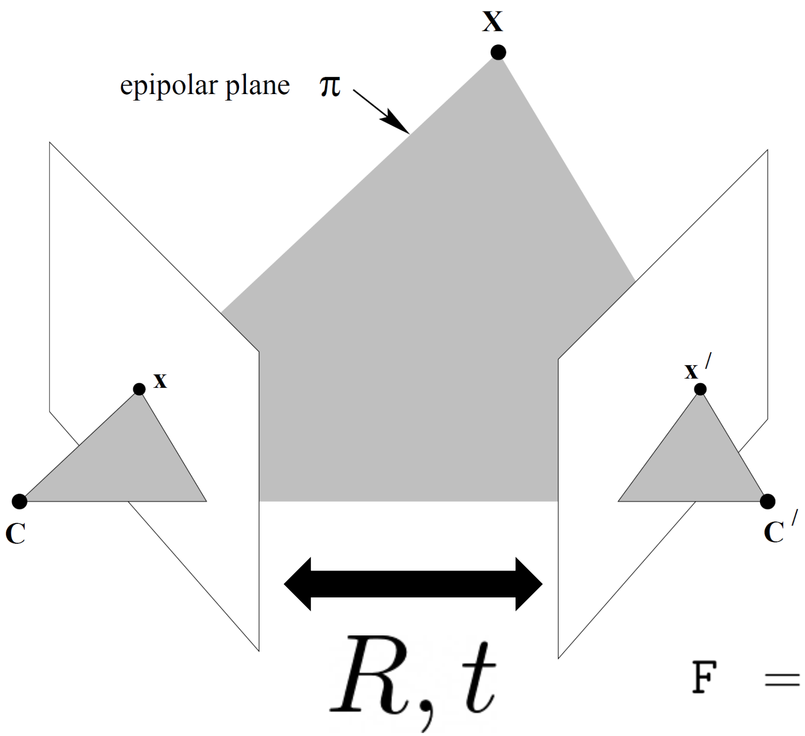
- Fundamental matrix

$$F = [\mathbf{e}']_{\times}P'P^{+}$$

3x3

$$\mathbf{l}' = F\mathbf{x}$$

Fundamental Matrix



$$F = [\mathbf{e}']_\times P' P^+ \qquad \mathbf{e}' = (P'C)$$

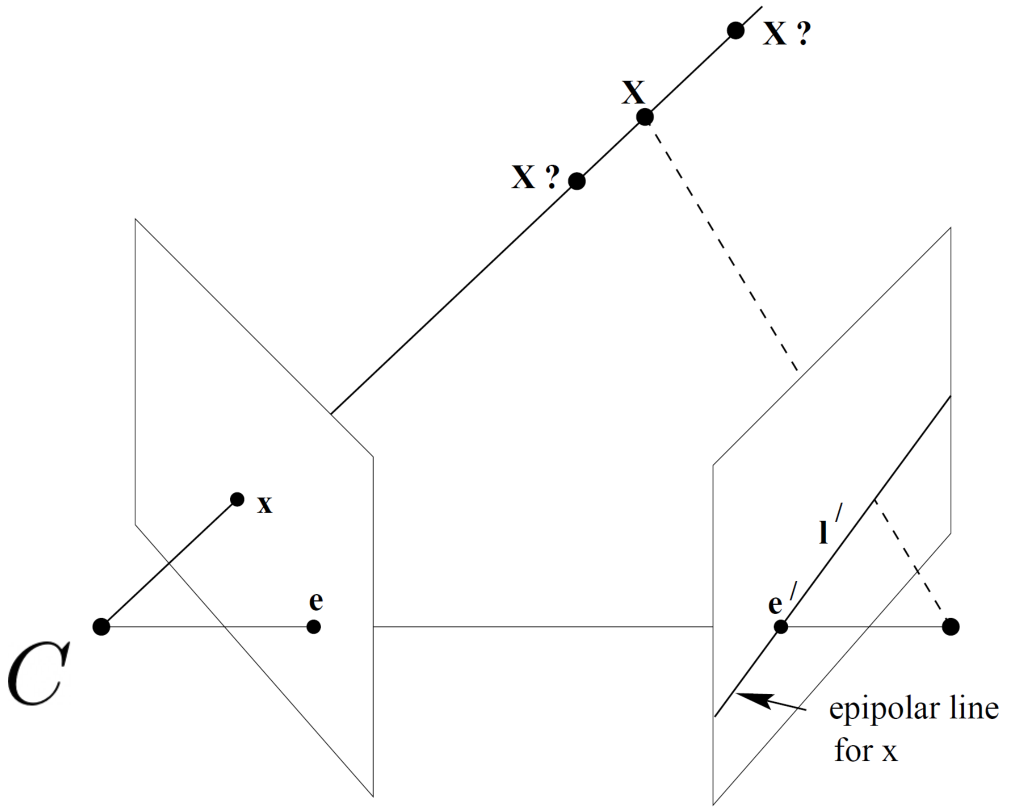
$$\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}] \qquad \mathtt{P}' = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}]$$

$$\mathtt{P}^+ = \begin{bmatrix} \mathtt{K}^{-1} \\ \mathbf{0}^\mathsf{T} \end{bmatrix} \qquad \mathbf{C} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

$$\begin{aligned} \mathtt{F} &= [\mathtt{P}'\mathbf{C}]_\times \mathtt{P}'\mathtt{P}^+ \\ &= [\mathtt{K}'\mathbf{t}]_\times \mathtt{K}'\mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\mathsf{T}}[\mathbf{t}]_\times \mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\mathsf{T}}\mathtt{R}[\mathtt{R}^\mathsf{T}\mathbf{t}]_\times \mathtt{K}^{-1} = \mathtt{K}'^{-\mathsf{T}}\mathtt{R}\mathtt{K}^\mathsf{T}[\mathtt{K}\mathtt{R}^\mathsf{T}\mathbf{t}]_\times \end{aligned}$$

$$\mathbf{e} = \mathtt{P}\begin{pmatrix} -\mathtt{R}^\mathsf{T}\mathbf{t} \\ 1 \end{pmatrix} = \mathtt{K}\mathtt{R}^\mathsf{T}\mathbf{t} \qquad \mathbf{e}' = \mathtt{P}'\begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \mathtt{K}'\mathbf{t} \qquad \mathtt{F} = [\mathbf{e}']_\times \mathtt{K}'\mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\mathsf{T}}[\mathbf{t}]_\times \mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\mathsf{T}}\mathtt{R}[\mathtt{R}^\mathsf{T}\mathbf{t}]_\times \mathtt{K}^{-1} = \mathtt{K}'^{-\mathsf{T}}\mathtt{R}\mathtt{K}^\mathsf{T}[\mathbf{e}]_\times$$

Properties of Fundamental Matrix



$\mathbf{x}'$ is on the epiploar line $\mathbf{l}' = F\mathbf{x}$

$$\mathbf{x}'^T F \mathbf{x} = 0$$

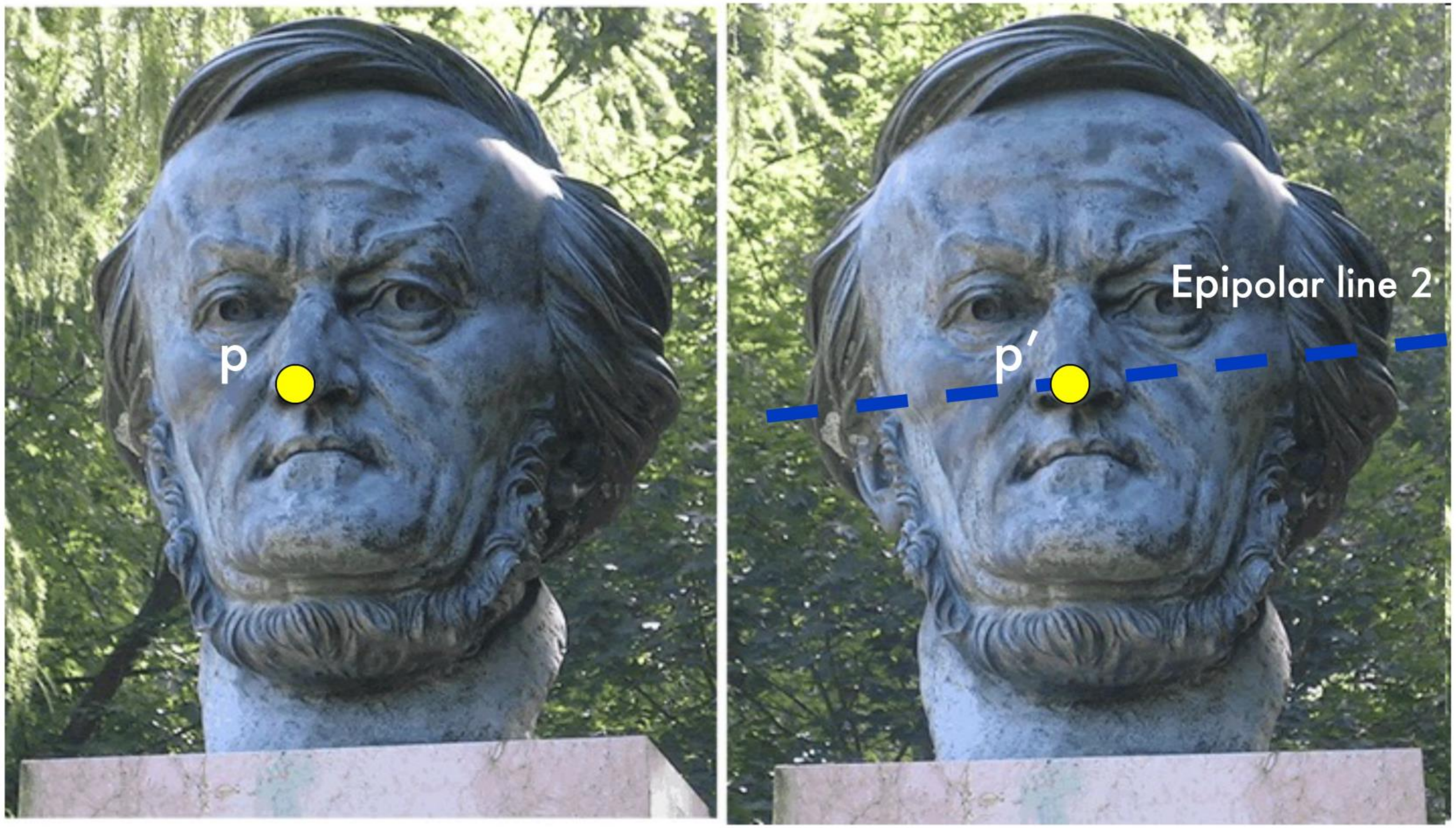
- Transpose: if F is the fundamental matrix of (P, P'), then F^T is the fundamental matrix of (P', P)
- Epipolar line: $\mathbf{l}' = F\mathbf{x}$ $\quad \mathbf{l} = F^T\mathbf{x}'$
- Epipole: $\mathrm{e}'^{\mathsf{T}}\mathrm{F} = \mathbf{0}$ $\quad \mathbf{Fe} = \mathbf{0}$

$$\mathrm{e}'^{\mathsf{T}}(\mathbf{Fx}) = (\mathrm{e}'^{\mathsf{T}}\mathrm{F})\mathbf{x} = 0 \text{ for all } \mathbf{x}$$

- 7 degrees of freedom

$$\det \mathbf{F} = 0$$

Why the Fundamental Matrix is Useful?



$$\mathbf{l}' = F\mathbf{p}$$

Further Reading

- Stanford CS231A: Computer Vision, From 3D Reconstruction to Recognition, Lecture 5 https://web.stanford.edu/class/cs231a/syllabus.html

- Multiview Geometry in Computer Vision, Richard Hartley and Andrew Zisserman, Chapter 9, Epipolar Geometry and Fundamental Matrix