The Physics of Virtual Worlds

CS 6334 Virtual Reality
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Review of VR Systems

INPUT
- Head Tracker
- Game Controller
- Keyboard & Mouse

COMPUTATION
- Virtual World Generator
  - Last lecture: Geometry
  - This lecture: Physics
- Visual Renderer
- Aural Renderer
- Haptic Renderer

OUTPUT
- Visual Display
- Aural Display
- Haptic Display

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The Geometry of Virtual Worlds

3D Rotation
3D Translation $R_x + t$

Object Coordinates

World Coordinates
The Physics of Virtual Worlds

Gravity

External force $F$
PyBullet Example
PyBullet Example

Credit: Xiangyun Meng at UW
Physics Simulation

• Dynamical system

State of the virtual world
• Object positions
• Object shapes
• Forces
• Energy
...

$S_t \rightarrow \text{Physics Engine} \rightarrow S_{t+1}$

Pendulum
Particle Dynamics

• Determine the states of particles (e.g., position)
Particle Dynamics

• Determine the position of a mass-less particle

• Given velocity field $\mathbf{v}(\mathbf{x}, t)$

• Initial Value Problem

$$x_p(0) = x_0$$

$$\frac{dx_p(t)}{dt} = \dot{x}_p(t) = \mathbf{v}(x_p, t)$$

How to calculate $x_p(t)$
Differential Equations

• A differential equation is an equation that relates one or more functions and their derivatives

\[
\frac{d\mathbf{x}_p(t)}{dt} = \dot{x}_p(t) = \mathbf{v}(\mathbf{x}_p, t)
\]

• Ordinary Differential Equation (ODE)
  • An equation that contains functions of only one independent variable and its derivatives
  • First-order ODE
Initial Value Problem

- Euler integration

\[
\frac{dx_p(t)}{dt} = \lim_{\epsilon \to 0} \frac{x_p(t + \epsilon) - x_p(t)}{\epsilon}
\]

\[
\frac{dx_p(t)}{dt} = \frac{x_p(t + \Delta t) - x_p(t)}{\Delta t}
\]

\[
x_p(t + \Delta t) - x_p(t)
\]

\[
\Delta t
\]

\[
x_p(t) + \Delta t \cdot \mathbf{v}(x_p, t)
\]

Position of the mass-less particle

\[
x_p(0) = x_0
\]

\[
\frac{dx_p(t)}{dt} = \dot{x}_p(t) = \mathbf{v}(x_p, t)
\]
Particle Dynamics

• Determine the position of a particle with mass
• Newton’s second law

\[ \mathbf{f} = m \mathbf{a} \]

Vector sum of all forces applied to each body in a system, newtons (N)

Vector acceleration of each body with respect to an inertial reference frame, m/sec²

Mass of the body, kg

Acceleration of gravity \( g = 9.81 \) m/sec²
Momentum

• The momentum of a body is

\[ p(t) = m v(t) \]

  Mass of the body, kg  Velocity of the body, m/sec

• Newton’s second law

\[ f(t) = \frac{d}{dt} p(t) = m \frac{d}{dt} v(t) = ma(t) \]
Newton’s Second Law

• Example

\[ ma = f_s + f_g + f_e \]

Bargteil, A., Shinar T. *An introduction to physics-based animation*, ACM SIGGRAPH 2018 Courses, 2018
A Particle with Mass

• Initial value problem

\[ x_p(0) = x_0 \]
\[ \frac{d^2 x_p(t)}{dt^2} = \ddot{x}_p(t) = \frac{f(x_p, t)}{m_p} \]

• First-order equations

\[ x_p(0) = x_0 \]
\[ v_p(0) = v_0 \]
\[ \frac{dx_p(t)}{dt} = \dot{x}_p(t) = v_p(t) \]
\[ \frac{dv_p(t)}{dt} = \dot{v}_p(t) = \frac{f(x_p, t)}{m_p} \]

Euler’s method

\[ v_p(t + \Delta t) = v_p(t) + \Delta t \cdot \frac{f(x_p, t)}{m_p} \]
\[ x_p(t + \Delta t) = x_p(t) + \Delta t \cdot v_p(t + \Delta t) \]
Materials

Rigid bodies
• No deformation

Soft bodies
• Deform elastically and plastically

Fluids
• Air, water, honey, etc.

Particles with springs
Rigid Bodies

• No deformation
• 6 DOF: 3D translation and 3D rotation
• Particles with very stiff springs
• Center of mass

\[ \mathbf{x}_{com} = \frac{\sum_{i=1}^{N} m_i \mathbf{p}_i}{\sum_{i=1}^{N} m_i} \]

Bargteil, A., Shinar T. An introduction to physics-based animation, ACM SIGGRAPH 2018 Courses, 2018
Object Space vs. World Space

(a) Object space.

(b) World space.

World position

\[ r(t) = R(t)r_0 \]

\[ p(t) = x(t) + r(t) \]

\[ p(t) = x(t) + R(t)r_0 \]
Linear Velocity

\[ p(t) = x(t) + R(t)r_0 \]

\[ v(t) = \dot{p}(t) = \dot{x}(t) + \dot{R}(t)r_0 \]

Linear velocity

- Motion of the particle due to linear velocity of the body
Instantaneous Rotation

\[ p(t) = x(t) + R(t)r_0 \]

\[ v(t) = \dot{p}(t) = \dot{x}(t) + \dot{R}(t)r_0 \]

Motion of the particle due to the instantaneous rotation of the body about its center of mass.
Angular Velocity $\omega$

Euler’s rotation theorem $\dot{\mathbf{R}}(t)\mathbf{r}_0$

- The vector whose direction is the instantaneous axis of rotation
- Length is the rate of rotation in radians per second

$$\dot{\mathbf{R}}(t)\mathbf{r}_0 = \mathbf{\omega}(t) \times \mathbf{r}(t)$$

$$\mathbf{v}(t) = \dot{\mathbf{p}}(t) = \dot{\mathbf{x}}(t) + \mathbf{\omega}(t) \times \mathbf{r}(t)$$

$$\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_0 \quad \dot{\mathbf{R}}(t) = \mathbf{\omega}(t) \times \mathbf{R}(t).$$
Linear Momentum

\[ P(t) = \sum_{i=1}^{N} m_i v_i(t) \]

\[ P(t) = \sum_{i=1}^{N} m_i (\dot{x}(t) + \omega(t) \times r_i(t)) \]

\[ = \sum_{i=1}^{N} m_i \dot{x}(t) + \omega(t) \times \left( \sum_{i=1}^{N} m_i r_i(t) \right) \]

\[ P(t) = M \dot{x}(t) \quad M = \sum_{i=1}^{N} m_i \]

(b) World space.

Derivation HW1
Angular Momentum

\[
L(t) = \sum_{i=1}^{N} r_i(t) \times m_i v_i(t)
\]

\[
L(t) = \sum_{i=1}^{N} m_i r_i(t) \times (\dot{x}(t) + \omega(t) \times r_i(t))
\]

\[
= \sum_{i=1}^{N} m_i r_i(t) \times \dot{x}(t) + \sum_{i=1}^{N} m_i r_i(t) \times \omega(t) \times r_i(t)
\]

\[
L(t) = \sum_{i=1}^{N} m_i r_i(t) \times (\omega(t) \times r_i(t))
\]
Angular Momentum

\[ L(t) = \sum_{i=1}^{N} m_i r_i(t) \times (\omega(t) \times r_i(t)) \]

\[ \omega \times r = -r \times \omega \]

\[ L(t) = \sum_{i=1}^{N} m_i r_i(t) \times (-r_i(t) \times \omega(t)) \]

Cross product matrix \( -r^* = r^* T \)

\[ r^* = \begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix} \]

\[ L(t) = \sum_{i=1}^{N} m_i r_i^*(t) (r_i^* T(t) \omega(t)) \]

\[ = \left( \sum_{i=1}^{N} m_i r_i^*(t) r_i^* T(t) \right) \omega(t) \]
Angular Momentum

\[ L(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^*(t) (\mathbf{r}_i^* T(t) \omega(t)) \]

\[ = \left( \sum_{i=1}^{N} m_i \mathbf{r}_i^*(t) \mathbf{r}_i^* T(t) \right) \omega(t) \]

Inertia tensor

\[ I(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^*(t) \mathbf{r}_i^* T(t) \]

\[ L(t) = I(t) \omega(t) \]

\[ I(t) = \sum_{i=1}^{N} m_i \mathbf{r}_i^*(t) \mathbf{r}_i^* T(t) \]

\[ = \sum_{i=1}^{N} m_i \left( \mathbf{r}_i^T \mathbf{r}_i \delta - \mathbf{r}_i \mathbf{r}_i^T \right) \]

\[ = \mathbf{R}(t) \sum_{i=1}^{N} m_i \left( \mathbf{r}_{0i}^T \mathbf{r}_{0i} \delta - \mathbf{r}_{0i} \mathbf{r}_{0i}^T \right) \mathbf{R}(t)^T \]

\[ = \mathbf{R}(t) I_0 \mathbf{R}(t)^T. \]

\[ \mathbf{r}^* \mathbf{r}^* T = \mathbf{r}^T \mathbf{r} \delta - \mathbf{r} \mathbf{r}^T \]

\( \delta \) is the 3 \times 3 identity matrix

\( \mathbf{r} = \mathbf{R} \mathbf{r}_0 \)
Force and Torque

Linear momentum \[ \mathbf{P}(t) = M \dot{\mathbf{x}}(t) \quad M = \sum_{i=1}^{N} m_i \]

Angular momentum \[ \mathbf{L}(t) = \mathbf{I}(t) \omega(t) \]

- Newton’s second law

\[ \frac{d}{dt} \begin{pmatrix} \mathbf{P}(t) \\ \mathbf{L}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{f}(t) \\ \mathbf{\tau}(t) \end{pmatrix} \]

- When a force apply to center of mass
  \[ \mathbf{a} = \mathbf{f} / M \]

- When a force apply to a point
  \[ \mathbf{\tau} = \mathbf{r} \times \mathbf{f} \]
Dynamics of Rigid Bodies

\[ \mathbf{v}(t) = \frac{\mathbf{P}(t)}{M} \]
\[ \mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_0\mathbf{R}(t)^T \]
\[ \boldsymbol{\omega}(t) = \mathbf{I}(t)^{-1}\mathbf{L}(t) \]

\[
\frac{d}{dt} \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ \boldsymbol{\omega}^\star(t)\mathbf{R}(t) \\ \mathbf{f}(t) \\ \tau(t) \end{pmatrix}
\]

- Linear Velocity
- Angular Velocity
- Force
- Torque
Rigid Body Simulation Examples

$S_t \xrightarrow{\text{Physics Engine}} S_{t+1}$

https://gfycat.com/
Further Readings

• Section 8.1, 8.3 in Virtual Reality, Steven LaValle
